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Anomalies and Differential Geometry I

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Universidad Autónoma de Madrid, PhD Course.

Plan of the course

* Generalities about anomalies
  - The axial anomaly
  - Gauge anomalies
  - A first contact with gravitational anomalies
* Functional methods. Wess-Zumino consistency conditions
* Anomalies and differential geometry
* Stora-Zumino descent chain
* Consistent vs. covariant anomalies

Bibliography (a sample)

Books:

* R.A. Bertlmann, "Anomalies in Quantum Field Theory", Oxford 1996
* B. Zumino, "Chiral Anomalies and Differential Geometry", in Relativity, Groups and Topology, Elsevier 1983.
* K. Fujikawa & H. Suzuki, "Path integrals and Quantum Anomalies", Oxford 2004
* L. Álvarez-Gaumé & M.A. Vázquez-Mozo, "Introduction to Anomalies", Springer (to appear)

General QFT books:

* L. Álvarez-Gaumé & M.A. Vázquez-Mozo, "An Invitation to Quantum Field Theory", Springer 2012 (Chapter 9)
* M.D. Schwartz, "Quantum Field Theory and the Standard Model", Cambridge 2014

Online Reviews:

* J.A. Harvey, "TASI Lectures on Anomalies", hep-th/0509097
* A. Bilal, "Lectures on Anomalies", hep-th/0802.0634.

Anomalies: a very quick introduction

An **anomaly** is the **quantum breaking** of a **classical symmetry**.

Anomalies can be of two very different kind:

The nice type

* They affect a ***nonfundamental symmetries***, e.g.

  - Scale invariance

  - Global symmetries

These anomalies are at the origin of very interesting physical phenomena:

asymptotic freedom, $\pi^0 \longrightarrow 2\gamma$, anomaly matching…

The nasty type

* They affect **local (gauge) invariances**, e.g.

  - Gauge anomalies

  - Gravitational anomalies

These are very dangerous anomalies that have to be **cancelled**.

Unphysical (ghost) states do not decouple and the whole theory becomes **inconsistent**.

“Good” anomalies

First example: Scale invariance

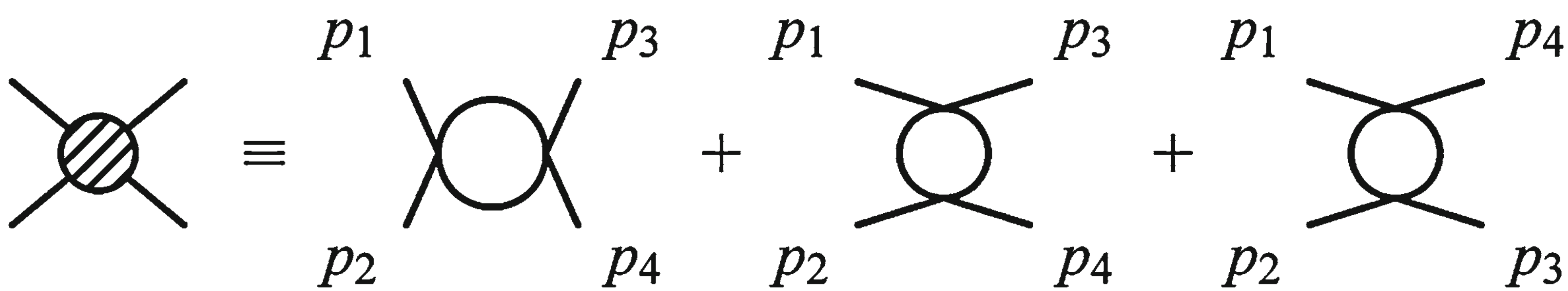
Consider a **massless** φ^4 theory: **classically** it is **scale invariant**:

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4!}\phi^4\right)$$

$$x^\mu \to \xi x^\mu,$$

$$\phi(x) \to \xi^{-1}\phi(\xi^{-1}x).$$

This invariance is broken by **quantum corrections**. Regularization and renormalization requires the introduction of an energy scale that breaks scale invariance. At one loop:



This results in the **running** of the coupling constant.

$$\beta(\lambda) = \frac{3\hbar\lambda^2}{16\pi^2} \quad \Longrightarrow \quad \lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3}{16\pi^3}\lambda(\mu_0)\log\left(\frac{\mu}{\mu_0}\right)}$$

so physics at different scales "does not look the same".

In **QCD** this quantum breaking of scale invariance is responsible for the most interesting features of the theory, such as **asymptotic freedom** and **confinement**.

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In **QCD** this quantum breaking of scale invariance is responsible for the most interesting features of the theory, such as **asymptotic freedom** and **confinement**.

Second example: The axial anomaly

Let us look at QED:

$$S_{\mathrm{QED}} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}A\!\!\!/\psi \right]$$

The theory has a $\mathrm{U}(1)$ **gauge invariance**

$$\psi(x) \longrightarrow e^{i\alpha}\psi(x), \qquad \overline{\psi}(x) \longrightarrow e^{-i\alpha}\overline{\psi}(x), \qquad \text{with} \qquad \alpha \in \mathbb{R}$$

with a **conserved vector current**

$$J^{\mu}_{\mathrm{V}} = \overline{\psi}\gamma^{\mu}\psi \qquad \Longrightarrow \qquad \partial_{\mu}J^{\mu}_{\mathrm{V}} = 0$$

This current couples to a **propagating gauge field** and its **invariance** is **crucial** for the internal consistency of the theory (e.g. unitarity).

In addition, we also have ***global* axial-vector transformations**

$$\psi(x) \longrightarrow e^{i\beta\gamma_5}\psi(x), \qquad \overline{\psi}(x) \longrightarrow \overline{\psi}(x)e^{i\beta\gamma_5}, \qquad \text{with} \qquad \beta \in \mathbb{R}$$

The associated conserved axial-vector current is **conserved** in the **massless limit**

$$J^\mu_{\mathrm{A}} = \overline{\psi}\gamma^\mu\gamma_5\psi \qquad \Longrightarrow \qquad \partial_\mu J^\mu_A = 2im\overline{\psi}\gamma_5\psi.$$

(pseudovector-pseudoscalar equivalence)

In the quantum theory, both the axial and the vector-axial current are **composite operators** that need to be **defined**.

The question is whether these operators can be defined to satisfy the **quantum conservation equations**

$$\partial_\mu\langle J^\mu_V(x)\rangle \stackrel{?}{=} 0$$

$$\partial_\mu\langle J^\mu_A(x)\rangle \stackrel{?}{=} 0$$

To analyze the problem, we look at a Dirac fermion coupled to an **classical external U(1) gauge field** $\mathscr{A}_\mu(x)$

$$S_{\rm int} = -e\int d^4x\, J^\mu_{\rm V}(x)\mathscr{A}_\mu(x)$$

(remember that $J^\mu_{\rm V} = \overline{\psi}\gamma^\mu\psi$)

The **expectation value** of the axial current in this background is given by

$$\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{\displaystyle\int\mathcal{D}\psi\mathcal{D}\overline{\psi}\, J^\mu_{\rm A}(x) e^{i\int d^4x\left[(i\partial\!\!\!/ - m)\psi - eJ^\mu_{\rm V}\mathscr{A}_\mu\right]}}{\displaystyle\int\mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{i\int d^4x\left[\overline{\psi}(i\partial\!\!\!/ - m)\psi - eJ^\mu_{\rm V}\mathscr{A}_\mu\right]}}.$$

This correlation function can be computed in perturbation theory

$$\begin{aligned}\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} &= -ie\int d^4y\langle 0|T[J^\mu_{\rm A}(x)J^\alpha_{\rm V}(y)]|0\rangle\mathscr{A}_\alpha(y) \\ &\quad - \frac{e^2}{2}\int d^4y_1 d^4y_2\langle 0|T[J^\mu_{\rm A}(x)J^\alpha_{\rm V}(y_1)J^\beta_{\rm V}(y_2)]|0\rangle\mathscr{A}_\alpha(y_1)\mathscr{A}_\beta(y_2) + \ldots\end{aligned}$$

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We are faced with the calculation of the following **free-field** correlation function

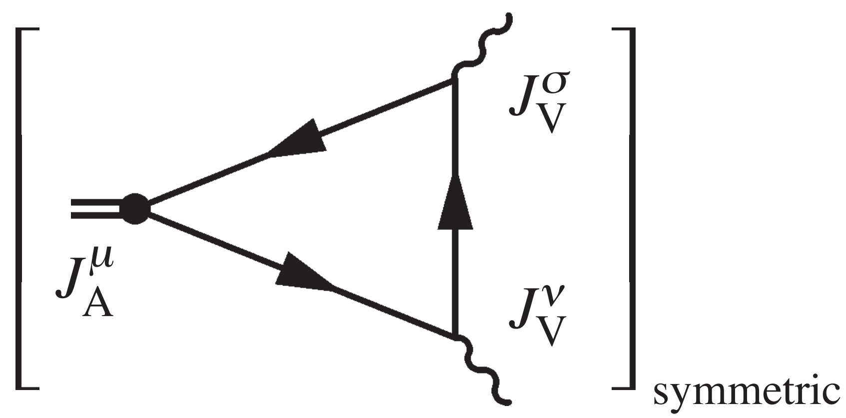
$$C^{\mu\nu\sigma}(x,y) = \langle 0|T[J_{\rm A}^{\mu}(x)J_{\rm V}^{\nu}(y)J_{\rm V}^{\sigma}(0)]|0\rangle$$

Which, applying **Wick's theorem** gives

$$C^{\mu\nu\sigma}(x,y) = \langle 0|\overline{\psi}\gamma^{\mu}\gamma_5\psi(x)\,\overline{\psi}\gamma^{\nu}\psi(y)\,\overline{\psi}\gamma^{\sigma}\psi(0)|0\rangle + \langle 0|\overline{\psi}\gamma^{\mu}\gamma_5\psi(x)\,\overline{\psi}\gamma^{\nu}\psi(y)\,\overline{\psi}\gamma^{\sigma}\psi(0)|0\rangle$$

These contractions are codified in the celebrated **triangle diagram**:

$$C^{\mu\nu\sigma}(x,y) =$$



The sought conservation equation is then

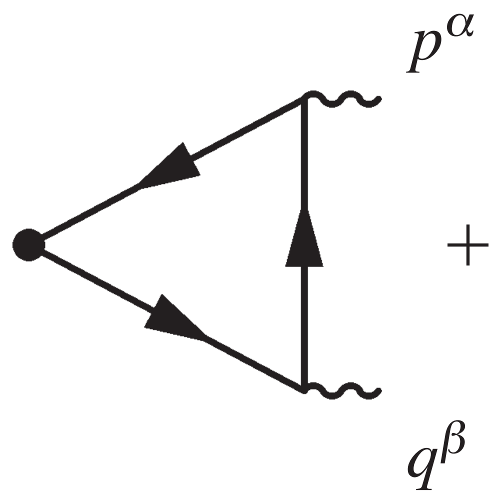
$$\partial_\mu\langle J_{\rm A}^{\mu}(x)\rangle_{\mathscr{A}} = -\frac{e^2}{2}\int d^4y_1 d^4y_2\,\partial_\mu^{(x)}C^{\mu\nu\sigma}(x,y_1-y_2)\mathscr{A}_\nu(y_1)\mathscr{A}_\sigma(y_2)$$

It is convenient to work in **momentum space**

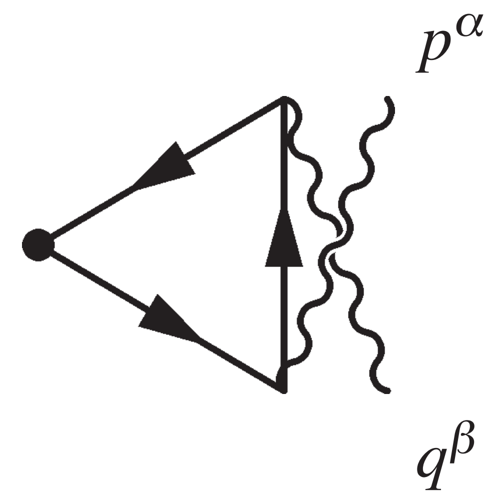
$$e^2\langle 0|T[J_{\mathrm{A}}^{\mu}(0)J_{\mathrm{V}}^{\alpha}(x_1)J_{\mathrm{V}}^{\beta}(x_2)]|0\rangle = \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}i\Gamma^{\mu\alpha\beta}(p,q)e^{ip\cdot x_1+iq\cdot x_2}$$

where

$$i\Gamma_{\mu\alpha\beta}(p,q) = (p+q)^{\mu}$$

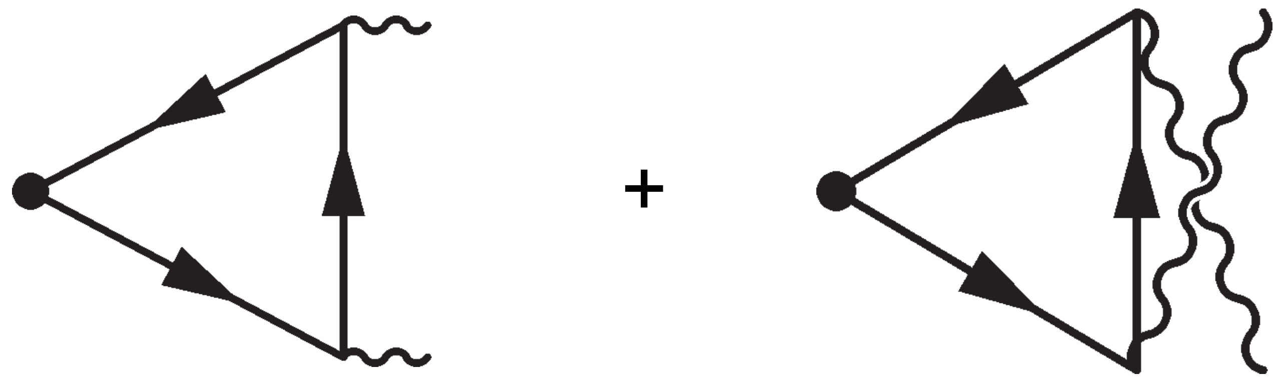


$$+ \quad (p+q)^{\mu}$$



and the **anomaly equation** to be computed is

$$(p+q)_{\mu}i\Gamma^{\mu\alpha\beta}(p,q) = ?$$



Applying the Feynman rules of QED, we have

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q) &= e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not{\ell}-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right) \\
&+ \left(\begin{array}{c} p\leftrightarrow q \\ \alpha\leftrightarrow\beta \end{array}\right).
\end{aligned}
$$

However, these integrals are **ambiguous**!

The problem is that they are **linearly divergent**.

Let us look at a simpler one-dimensional integral

$$I(a) = \int_{-\infty}^{\infty} dx\, f(x+a) \quad \begin{cases} \lim\limits_{|x|\to\infty} f(x) = \text{constant} & \Longrightarrow \quad \text{linearly divergent} \\ \lim\limits_{|x|\to\infty} f(x) = 0 & \Longrightarrow \quad \text{logarithmically divergent or convergent} \end{cases}$$

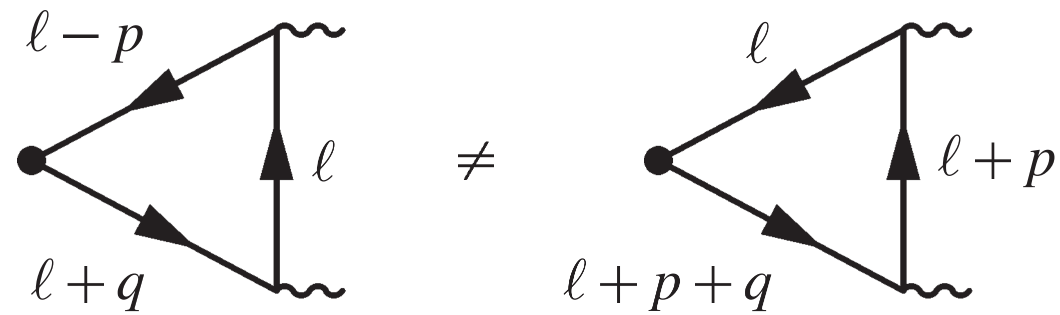
Naively, $I(a) = I(0)$. However, computing the derivative

$$I'(a) = \int_{-\infty}^{\infty} dx\, f'(x+a) = f(\infty) - f(-\infty) \quad \begin{cases} \neq 0 & \text{if linearly divergent} \\ = 0 & \text{if logarithmically divergent or convergent} \end{cases}$$

Hence, if the integral is linearly divergent the **result** of the integration depends on a **shift** in the integration variable!

The same happens for multidimensional integrals.

Thus, the triangle diagram is **ambiguous** because its contribution depends on how we **label** the loop momentum!



From **Lorentz invariance**, the most general form of $i\Gamma_{\mu\alpha\beta}(p,q)$ is (the **Levi-Civita tensor** is due to γ_5)

$$\begin{aligned} i\Gamma_{\mu\alpha\beta}(p,q) &= f_1\varepsilon_{\mu\alpha\beta\sigma}p^\sigma + f_2\varepsilon_{\mu\alpha\beta\sigma}q^\sigma + f_3\varepsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda \\ &+ f_4\varepsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda + f_5\varepsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda \\ &+ f_6\varepsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda + f_7\varepsilon_{\alpha\beta\sigma\lambda}p_\mu p^\sigma q^\lambda + f_8\varepsilon_{\alpha\beta\sigma\lambda}q_\mu p^\sigma q^\lambda \end{aligned}$$

From **Bose symmetry**

$$i\Gamma_{\mu\alpha\beta}(p,q) = i\Gamma_{\mu\beta\alpha}(q,p) \quad \Longrightarrow \quad \begin{aligned} f_1(p,q) &= -f_2(q,p), & f_3(p,q) &= -f_6(q,p), \\ f_4(p,q) &= -f_5(q,p), & f_7(p,q) &= -f_8(q,p). \end{aligned}$$

A bit of dimensional analysis: $[\Gamma_{\mu\alpha\beta}] = \text{energy}$

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q) = {} & f_1 \varepsilon_{\mu\alpha\beta\sigma} p^\sigma + f_2 \varepsilon_{\mu\alpha\beta\sigma} q^\sigma + f_3 \varepsilon_{\mu\alpha\sigma\lambda} p_\beta p^\sigma q^\lambda \\
& + f_4 \varepsilon_{\mu\alpha\sigma\lambda} q_\beta p^\sigma q^\lambda + f_5 \varepsilon_{\mu\beta\sigma\lambda} p_\alpha p^\sigma q^\lambda \\
& + f_6 \varepsilon_{\mu\beta\sigma\lambda} q_\alpha p^\sigma q^\lambda + f_7 \varepsilon_{\alpha\beta\sigma\lambda} p_\mu p^\sigma q^\lambda + f_8 \varepsilon_{\alpha\beta\sigma\lambda} q_\mu p^\sigma q^\lambda
\end{aligned}
$$

Dimensions = $(\text{energy})^0$

Dimensions = $(\text{energy})^{-2}$

Thus, only $f_1(p,q)$ and $f_2(p,q)$ are (logarithmically) **divergent** and their values depend on the regularization scheme used.

The remaining integrals $f_3(p,q)$ to $f_8(p,q)$ are **convergent** and free of ambiguities.

Is there a **wise way of fixing** these regularization ambiguities?

So far we have ignored the issue of **gauge invariance**. The relevant gauge **Ward identities** reads

$$i\Gamma_{\mu\alpha\beta}(p,q) \Longrightarrow \begin{cases} p^{\alpha} i\Gamma_{\mu\alpha\beta}(p,q) = 0 \\ q^{\beta} i\Gamma_{\mu\alpha\beta}(p,q) = 0 \end{cases}$$

$$\mu \to J_A^{\mu}, \quad \alpha \to J_V^{\alpha}, \quad \beta \to J_V^{\beta}$$

Vector current conservation further constraints the functions $f_i(p,q)$

$$p^{\alpha} i\Gamma_{\mu\alpha\beta}(p,q) = \left(f_2 - p^2 f_5 - p\cdot q f_6\right)\varepsilon_{\mu\beta\alpha\sigma} q^{\alpha} p^{\sigma} \Longrightarrow f_2(p,q) = p^2 f_5(p,q) + p\cdot q f_6(p,q)$$

$$q^{\beta} i\Gamma_{\mu\alpha\beta}(p,q) = \left(f_1 - q^2 f_4 - p\cdot q f_3\right)\varepsilon_{\mu\alpha\beta\sigma} q^{\beta} p^{\sigma} \Longrightarrow f_1(p,q) = q^2 f_4(p,q) - p\cdot q f_3(p,q)$$

Hence, **gauge invariance completely fixes the ambiguities** and the anomaly is completely determined by **finite integrals**

$$(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) = \Big[p^2(f_5+f_7) + q^2(-f_4+f_8) + p\cdot q(-f_3+f_6+f_7+f_8)\Big]\varepsilon_{\alpha\beta\sigma\lambda} q^{\sigma} p^{\lambda}$$

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& + f_4 \varepsilon_{\mu\alpha\sigma\lambda} q_\beta p^\sigma q^\lambda + f_5 \varepsilon_{\mu\beta\sigma\lambda} p_\alpha p^\sigma q^\lambda \\
& + f_6 \varepsilon_{\mu\beta\sigma\lambda} q_\alpha p^\sigma q^\lambda + f_7 \varepsilon_{\alpha\beta\sigma\lambda} p_\mu p^\sigma q^\lambda + f_8 \varepsilon_{\alpha\beta\sigma\lambda} q_\mu p^\sigma q^\lambda
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$$

$$
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$$

Now we only have to evaluate the integrals

$$f_3(p,q) = -f_6(q,p) = \frac{ie^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{xy}{x(1-x)p^2 + y(1-y)q^2 + 2xy p\cdot q + m^2},$$

$$f_4(p,q) = -f_5(q,p) = \frac{ie^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{y(1-y)}{x(1-x)p^2 + y(1-y)q^2 + 2xy p\cdot q + m^2},$$

$$f_7(p,q) = -f_8(q,p) = 0,$$

Steven Adler
(b. 1939)

to find the result

$$(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) \;=\; \frac{ie^2}{2\pi^2}\epsilon_{\alpha\beta\sigma\nu}p^\sigma q^\nu + 2mi\Gamma_{\alpha\beta}(p,q)$$

John S. Bell
(1928-1990)

Back in position space, we get the famous **Adler-Bell-Jackiw anomaly**

Jack Steinberger
(b. 1921)

$$\boxed{\partial_\mu \langle J^\mu_A(x)\rangle_{\mathscr{A}} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta} + 2im\langle\overline{\psi}(x)\gamma_5\psi(x)\rangle_{\mathscr{A}}}$$

Roman Jackiw
(b. 1939)

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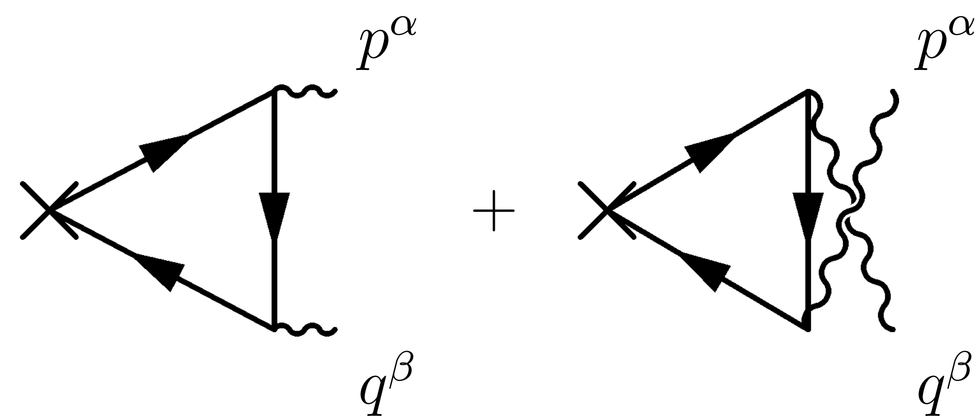
$i\Gamma_{\mu\nu}(p,q) \equiv$ 

$\times \equiv 2m\gamma_5$

Steven Adler
(b. 1939)

to find the result

$$(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) = \frac{ie^2}{2\pi^2}\epsilon_{\alpha\beta\sigma\nu}p^\sigma q^\nu + 2mi\Gamma_{\alpha\beta}(p,q)$$

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Roman Jackiw
(b. 1939)

Actually, there are other **choices**… Suppose we **shift** the loop momentum:

$$\ell^\mu \longrightarrow \ell^\mu + \alpha p^\mu + \beta q^\mu$$

$$i\Gamma_{\mu\alpha\beta}(p,q) \longrightarrow i\Gamma_{\mu\alpha\beta}(p,q) + \Delta_{\mu\alpha\beta}(\alpha,\beta)$$

- Parity
- Lorentz invariance
- Bose symmetry

$$i\Gamma_{\mu\alpha\beta}(p,q) \longrightarrow i\Gamma_{\mu\alpha\beta}(p,q) + \frac{ie^2}{8\pi^2} a \epsilon_{\mu\alpha\beta\sigma}(p-q)^\sigma$$

$$a = a(\alpha,\beta)$$

The **vector** and **axial Ward identities** now read:

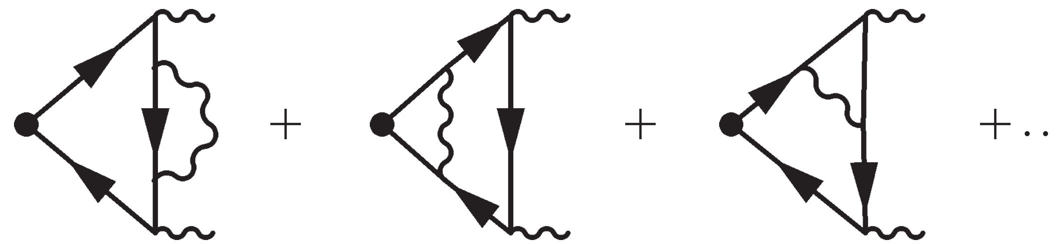
$$\begin{aligned}
(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) &= \frac{ie^2}{4\pi^2}(1-a)\epsilon_{\alpha\beta\sigma\nu}p^\sigma q^\nu + 2mi\Gamma_{\alpha\beta}(p,q), \\
p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= -\frac{ie^2}{8\pi^2}(1+a)\epsilon_{\alpha\beta\sigma\nu}p^\sigma q^\nu .
\end{aligned}$$

There is **no value of *a*** for which **both** are **preserved**!

our physical choice:

$$a = -1$$

Is the one-loop result enough? We can look at contributions of higher loop diagrams to the anomaly, e.g.



These diagrams contain **five** fermion propagator. The integration over the "triangle momentum" has the structure

$$\cdots \int \frac{d^4\ell}{(2\pi)^4}\prod_{i=1}^{5}\frac{i}{\not{\ell}+\not{\Delta}_i+i\varepsilon}\cdots$$

Steven Adler
(b. 1939)

and it is **unambiguous**. The integration over the **photon momentum** can be regularized in a gauge-invariant way, for example adding the term

$$\Delta S=\frac{1}{\Lambda^2}\int d^4x\, F_{\mu\nu}\Box F^{\mu\nu} \quad\Longrightarrow\quad G_{\mu\nu}(p)\sim\frac{\Lambda^2}{p^4}$$

William A. Bardeen
(b. 1941)

Hence, higher-loop triangles ***do not contribute*** to the anomaly.

This is known as the ***Adler-Bardeen theorem***

Instead of QED, we consider now a fermion coupled (in a certain representation) to an **external non-Abelian** gauge field

$$S = \int d^4x \left(i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi + g\overline{\psi}T^a_{\mathbf{R}}\gamma^\mu\psi\mathscr{A}^a_\mu \right)$$

Classically, the gauge current $J^{\mu a}_{\mathrm{V}} = \overline{\psi}\gamma^\mu T^a_{\mathbf{R}}\psi$ satisfies the conservation equation

$$(\mathcal{D}_\mu J^\mu_{\mathrm{V}})^a = 0 \quad \longrightarrow \quad \partial_\mu J^{\mu a}_{\mathrm{V}} + g f^{abc}\mathscr{A}^b_\mu J^{\mu c}_{\mathrm{V}} = 0$$

In addition we also have **global axial transformations**

$$\psi \longrightarrow e^{i\beta\gamma_5}\psi \qquad\qquad \overline{\psi} \longrightarrow \overline{\psi}e^{i\beta\gamma_5}$$

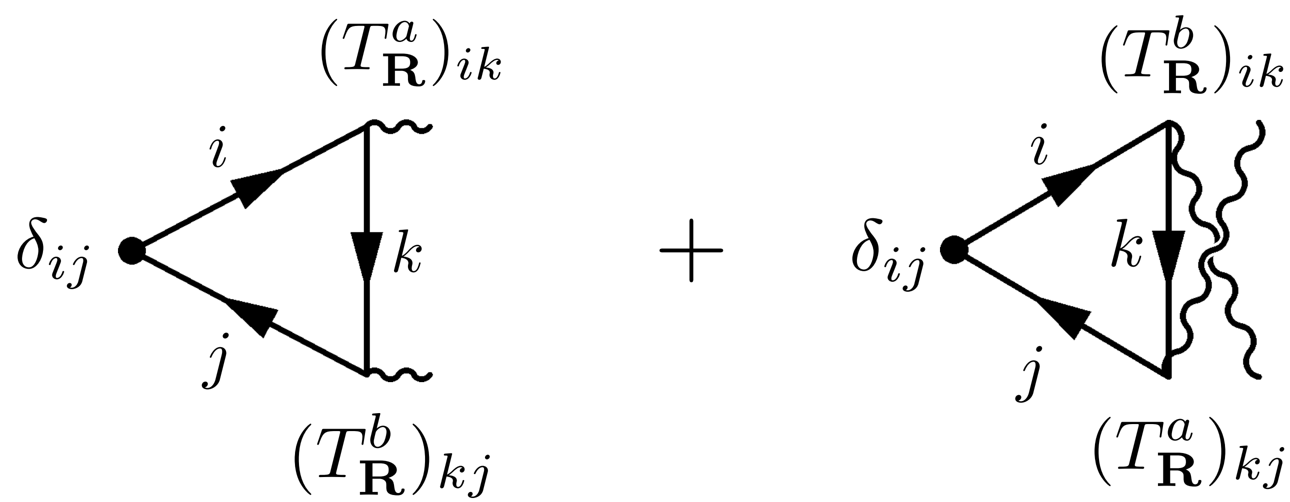
while its associated **singlet** axial current $J^\mu_{\mathrm{A}} = \overline{\psi}\gamma^\mu\gamma_5\psi$ satisfies the identity

$$\partial_\mu J^\mu_{\mathrm{A}} = 2im\overline{\psi}\psi$$

Similarly to QED, the calculation of the axial anomaly boils down to computing

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -\frac{g^2}{2}\int d^4y_1 d^4y_2 \partial^{(x)}_\mu \langle 0|T[J^\mu_{\rm A}(x) J^{\alpha a}_{\rm V}(y_1) J^{\beta b}_{\rm V}(y_2)]|0\rangle \mathscr{A}^a_\alpha(y_1)\mathscr{A}^b_\beta(y_2)+\ldots$$

Diagrammatically, we have again two triangle diagrams, these time with gauge group generators on the "vector" vertices



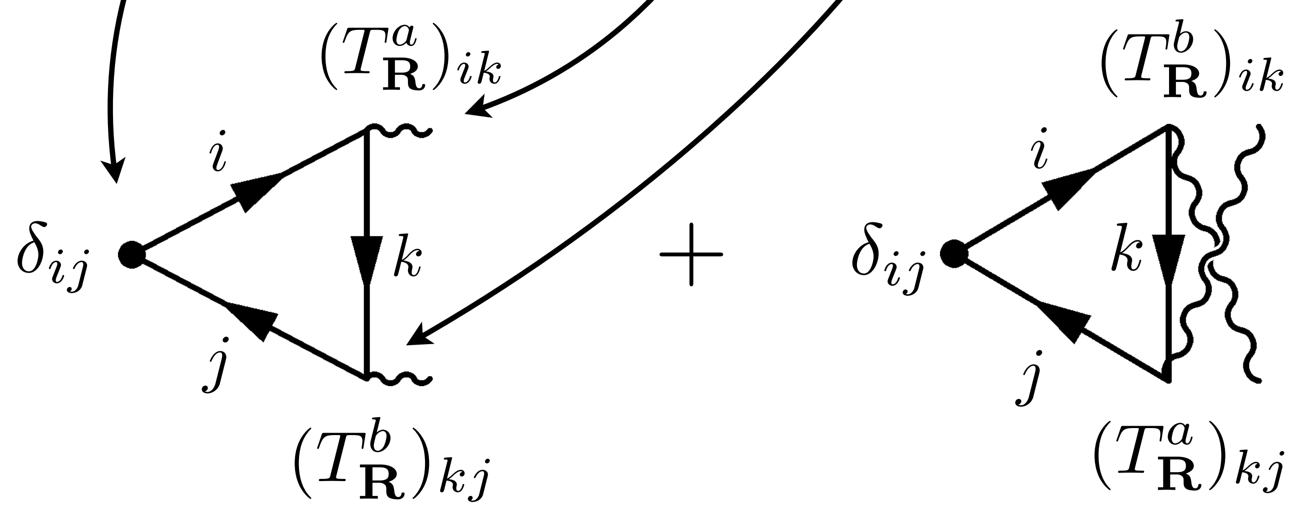
The two diagrams share the same gauge factor

$$\mathrm{Tr}\,(T^a_{\mathbf{R}} T^b_{\mathbf{R}}) = \mathrm{Tr}\,(T^b_{\mathbf{R}} T^a_{\mathbf{R}})$$

Similarly to QED, the calculation of the axial anomaly boils down to computing

$$\partial_\mu \langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = -\frac{g^2}{2}\int d^4y_1 d^4y_2 \partial^{(x)}_\mu \langle 0|T[J^\mu_{\mathrm{A}}(x) J^{\alpha a}_{\mathrm{V}}(y_1) J^{\beta b}_{\mathrm{V}}(y_2)]|0\rangle \mathscr{A}^a_\alpha(y_1)\mathscr{A}^b_\beta(y_2) + \ldots$$

Diagrammatically, we have again two triangle diagrams, these time with gauge group generators on the "vector" vertices



The two diagrams share the same gauge factor

$$\mathrm{Tr}\,(T^a_{\mathbf{R}} T^b_{\mathbf{R}}) = \mathrm{Tr}\,(T^b_{\mathbf{R}} T^a_{\mathbf{R}})$$

$$\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = -\frac{g^2}{2}\int d^4y_1 d^4y_2 \partial_\mu^{(x)}\langle 0|T[J_{\rm A}^\mu(x)J_{\rm V}^{\alpha a}(y_1)J_{\rm V}^{\beta b}(y_2)]|0\rangle \mathscr{A}_\alpha^a(y_1)\mathscr{A}_\beta^b(y_2)+\ldots$$

The rest of the calculation is identical to the case of QED. In momentum space, we get

$$(p+q)^\mu i\Gamma^{ab}_{\mu\alpha\beta}(p,q) = \frac{ig^2}{2\pi^2}\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}})\epsilon_{\alpha\beta\sigma\nu}p^\sigma q^\nu + 2mi\Gamma^{ab}_{\alpha\beta}(p,q)$$

Adding the external gauge fields and Fourier transforming back to position space, this leads to

$$\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}})\partial_\mu\mathscr{A}_\nu^a\partial_\alpha\mathscr{A}_\beta^b = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}})\partial_\mu\Big(\mathscr{A}_\nu^a\partial_\alpha\mathscr{A}_\beta^b\Big)$$

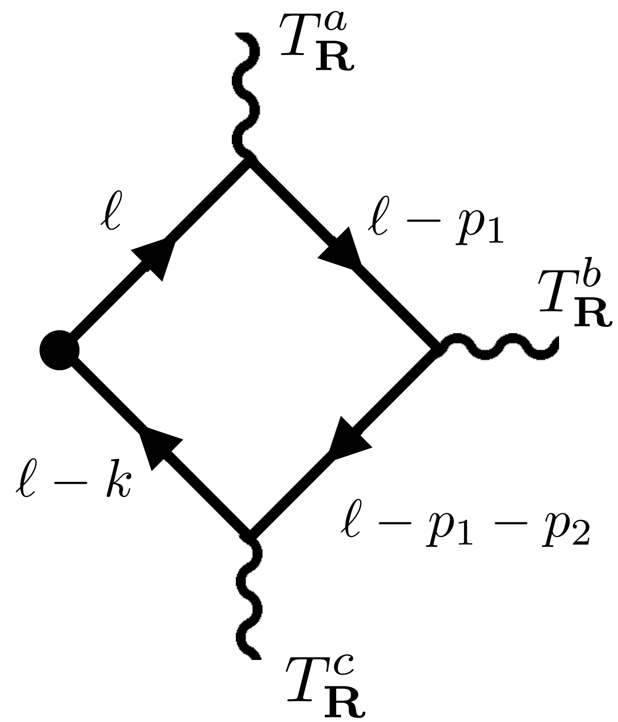
$$\Downarrow$$

$$\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu\mathrm{Tr}\,\Big(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta\Big)$$

The problem with this result is that it is **not gauge invariant**!

In fact, in the case of the singlet anomaly the triangle diagram is not enough.

We need to compute the **box diagrams** as well:



$+$ permutations

This gives a second **contribution cubic in the gauge fields** that **adds** up to the **triangle** result

$$\partial_\mu \langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta + \frac{2}{3}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right)$$

singlet anomaly

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta + \frac{2}{3}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right)$$

Here we identify the **Chern-Simons form**,

$$\epsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta + \frac{2}{3}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right) = \frac{1}{4}\mathrm{Tr}\left(\mathscr{F}^{\mu\nu}\widetilde{\mathscr{F}}_{\mu\nu}\right)$$

so the singlet anomaly can be written as

$$\boxed{\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\right)}$$

which is **gauge invariant**.

Important: although there is **contribution** to the anomaly from the **box** diagram, its **coefficient** is determined by the **triangle diagram**

Gauge anomalies

Prelude: quantum symmetries vs. gauge invariance

Wigner's theorem says that **global symmetries** are implemented on the Hilbert space by **unitary** or **antiunitary** operators:

$$\mathcal{U}(\alpha_i)|\psi\rangle = |\psi'\rangle \qquad \text{where, generically} \qquad |\psi\rangle \neq |\psi'\rangle$$

Look, for **example**, at the hydrogen atom: a $\mathrm{SO}(3)$ rotation acts on a state as

$$\mathcal{U}(\theta,\varphi,\psi)|n,j,m\rangle = \sum_{m'=-j}^{j} \mathscr{D}^{(j)}_{mm'}(\theta,\varphi,\psi)|n,j,m'\rangle$$

Gauge invariance is very different from this. In a gauge theory, a physical state is represented by **infinitely many rays** in the Hilbert space.

The space of physical states is smaller than the "naive" Hilbert space of the theory

$$\mathscr{H}_{\text{phys}} = \mathscr{H}/\mathscr{G}$$

Thus, **gauge invariance is not a symmetry but a redundancy**. Just a **technicality** to describe a spin-1 (or spin-2) theory in a way compatible with **locality** and **Lorentz invariance**.

But some of these redundant states have negative norm, e.g.

$$|\Psi\rangle = A_0|\Omega\rangle \longrightarrow \langle\Psi|\Psi\rangle < 0$$

These **dangerous** redundant states are **eliminated** from the physical spectrum by demanding **gauge invariance**:

$$\delta_{\text{gauge}}|\psi\rangle_{\text{phys}} = 0$$

Since $\delta_{\text{gauge}}A_0 = \dot{\epsilon}(x)$ we have

$$\delta_{\text{gauge}}|\Psi\rangle \neq 0 \longrightarrow |\Psi\rangle \text{ is not a physical state}$$

The **absence of ghosts** is preserved in time when the **quantum** theory is **gauge invariant**

$$[\delta_{\rm gauge}, H] = 0$$

This guarantees that

$$\delta_{\rm gauge}|\psi(0)\rangle = 0 \longrightarrow \delta_{\rm gauge}|\psi(t)\rangle = 0$$

i.e., the time evolution of a physical state is a physical state.

When **gauge invariance is anomalous**, ghosts can pop up

$\Downarrow$

the theory becomes **nonunitary**

$\Downarrow$

gauge anomalies should be **cancelled** in physical theories at all cost

Where can we expect gauge anomalies?

Chiral anomalies can only emerge in **even-dimensional** theories. Besides, parity **reverses** fermion helicity

$$\mathscr{P} : \psi_{R,L} \longrightarrow \psi_{L,R}$$

Thus, a parity-invariant theory contains as many right- and left-handed fermions in the same representation.

In this case, we can build **gauge-invariant mass terms** and regularize the theory using **Pauli-Villars** fields which preserve gauge invariance.

Gauge anomalies can arise only in **parity-violating** theories.

For example, consider N Dirac fermions with charges Q_i **chirally coupled** to an external **U(1) gauge field**

$$S=\sum_{i=j}^{N}\int d^4x\left[i\overline{\psi}_j\gamma^\mu\partial_\mu\psi_j+Q_i\overline{\psi}_j\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\psi_j\mathscr{A}_\mu\right]$$

This theory has a gauge symmetry

$$\psi_j(x)\longrightarrow\frac{1+\gamma_5}{2}\psi_j(x)+e^{iQ_j\alpha(x)}\frac{1-\gamma_5}{2}\psi_j(x)$$

$$\mathscr{A}_\mu(x)\longrightarrow\mathscr{A}_\mu(x)+\partial_\mu\alpha(x)$$

where the associated conserved current is of the V-A type

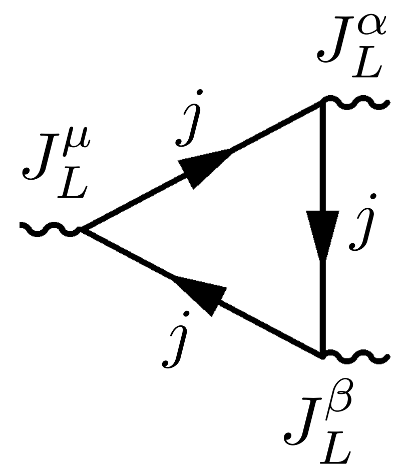
$$J^\mu_L=\sum_{j=1}^{N}Q_j\overline{\psi}_j\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\psi_j\qquad\text{with}\qquad\partial_\mu J^\mu_L=0$$

To **spot** the **gauge anomaly**, we have to compute

$$\partial_\mu \langle J_L^\mu(x)\rangle_{\mathscr{A}} = -\frac{1}{2}\int d^4y_1 d^4y_2 \langle 0|T[J_L^\mu(x) J_L^\alpha(y_1) J_L^\beta(y_2)]|0\rangle \mathscr{A}_\alpha(y_1)\mathscr{A}_\beta(y_2)$$

We have evaluate a **triangle** diagram with three **left currents** at the vertices

$$\sum_{j=1}^{N}$$



(summing over all fermion species in the loop)

and impose **Bose symmetry** on **all three vertices**

Even before computing it, we see that the result should be proportional to the quantity

$$\partial_\mu \langle J_L^\mu\rangle_{\mathscr{A}} \sim \sum_{j=1}^{N} Q_j^3 \qquad \text{which cancels if} \qquad \sum_{j=1}^{N} Q_j^3 = 0$$

A similar calculation for a **right-handed theory**

$$S=\sum_{i=j}^{N}\int d^4x\left[i\overline{\psi}_j\gamma^\mu\partial_\mu\psi_j+\widetilde{Q}_i\overline{\psi}_j\gamma^\mu\left(\frac{1+\gamma_5}{2}\right)\psi_j\mathscr{A}_\mu\right]$$

and we have

$$\partial_\mu\langle J_R^\mu\rangle_{\mathscr{A}}\sim-\sum_{j=1}^{N}\widetilde{Q}_j^3 \quad \text{which again cancels when} \quad \sum_{j=1}^{N}\widetilde{Q}_j^3=0$$

For a theory with N_R right-handed and N_L left-handed fermions, the cancellation condition for the anomaly reads

$$\sum_{j=1}^{N_R}\widetilde{Q}_j^3-\sum_{j=1}^{N_L}Q_j^3=0$$

We analyze now the **non-Abelian** case

$$S=\int d^4x\,\left[i\overline{\psi}\gamma^\mu\Big(\partial_\mu-i\mathscr{L}_\mu\Big)\left(\frac{1-\gamma_5}{2}\right)\psi+i\overline{\psi}\gamma^\mu\Big(\partial_\mu-i\mathscr{R}_\mu\Big)\left(\frac{1+\gamma_5}{2}\right)\psi\right]$$

where we have introduced **external gauge fields** coupled respectively to the **right-** and **left-handed component** of the fermion

$$\mathscr{L}_\mu(x)=\mathscr{L}_\mu^a(x)T^a \qquad\qquad \mathscr{R}_\mu(x)=\mathscr{R}_\mu^a(x)T^a$$

This theory has a $G_L\times G_R$ **gauge invariance**

$$\psi(x)\longrightarrow e^{iu_L^a(x)T^a}\left(\frac{1-\gamma_5}{2}\right)\psi(x)+e^{iu_R^a(x)T^a}\left(\frac{1+\gamma_5}{2}\right)\psi(x)$$

$$\mathscr{L}_\mu(x)\longrightarrow ie^{iu_L^a(x)T^a}\partial_\mu e^{-iu_L^a(x)T^a}+e^{iu_L^a(x)T^a}\mathscr{L}_\mu(x)e^{-iu_L^a(x)T^a}$$

$$\mathscr{R}_\mu(x)\longrightarrow ie^{iu_R^a(x)T^a}\partial_\mu e^{-iu_R^a(x)T^a}+e^{iu_R^a(x)T^a}\mathscr{R}_\mu(x)e^{-iu_R^a(x)T^a}$$

Alternatively, we can write the theory in terms of **vector** and **axial-vector gauge** fields

$$S=\int d^4x\Big[i\overline{\psi}\gamma^\mu\Big(\partial_\mu-i\mathscr{V}_\mu-i\mathscr{A}_\mu\gamma_5\Big)\psi\Big]$$

where $\mathscr{V}_\mu=\mathscr{V}_\mu^aT^a$ and $\mathscr{A}_\mu=\mathscr{A}_\mu^aT^a$ are given by

$$\mathscr{V}_\mu=\frac{1}{2}\Big(\mathscr{L}_\mu+\mathscr{R}_\mu\Big)\qquad\qquad \mathscr{A}_\mu=\frac{1}{2}\Big(\mathscr{L}_\mu-\mathscr{R}_\mu\Big)$$

In terms of these fields, we have **vector and axial gauge transformations**

$$\begin{aligned}\psi(x)&\longrightarrow e^{i\alpha^a(x)T^a}\psi(x)\\ \mathscr{V}_\mu(x)&\longrightarrow ie^{i\alpha^a(x)T^a}\partial_\mu e^{-i\alpha^a(x)T^a}\\ &\qquad+e^{i\alpha^a(x)T^a}\mathscr{V}_\mu(x)e^{-i\alpha^a(x)T^a}\\ \mathscr{A}_\mu(x)&\longrightarrow e^{i\alpha^a(x)T^a}\mathscr{A}_\mu(x)e^{-i\alpha^a(x)T^a}\end{aligned}$$

$$\begin{aligned}\psi(x)&\longrightarrow e^{i\beta^a(x)T^a\gamma_5}\psi(x)\\ \mathscr{V}_\mu(x)&\longrightarrow e^{i\beta^a(x)T^a}\mathscr{V}_\mu(x)e^{-i\beta^a(x)T^a}\\ \mathscr{A}_\mu(x)&\longrightarrow ie^{i\beta^a(x)T^a}\partial_\mu e^{-i\beta^a(x)T^a}\\ &\qquad+e^{i\beta^a(x)T^a}\mathscr{A}_\mu(x)e^{-i\beta^a(x)T^a}\end{aligned}$$

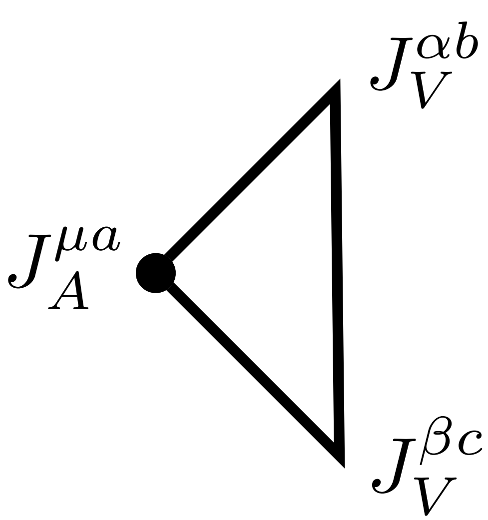
The classical conservation equations for the vector and axial-vector currents are

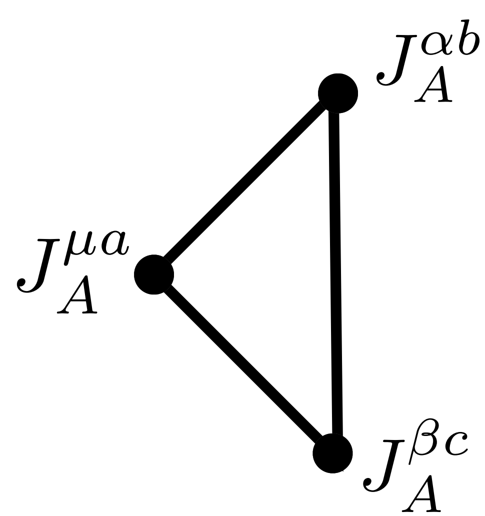
$$
(\mathcal{D}_\mu J^\mu_{\rm A})^a = 0 \quad \longrightarrow \quad
\begin{aligned}
\partial_\mu J^{\mu a}_{\rm A} + f^{abc}\mathscr{V}^a_\mu J^{\mu c}_{\rm A} + f^{abc}\mathscr{A}^b_\mu J^{\mu c}_{\rm A} &= 0 \\
(D_\mu J^\mu_{\rm A})^a + f^{abc}\mathscr{A}^b_\mu J^{\mu c}_{\rm A} &= 0
\end{aligned}
$$

To find the anomaly we have to calculate

$$
\langle(\mathcal{D}_\mu J^\mu_{\rm A})^a\rangle_{\mathscr{V},\mathscr{A}} = \frac{1}{Z}\int \mathscr{D}\psi\mathscr{D}\overline{\psi}\Big(\partial_\mu J^{\mu a}_{\rm A} + f^{abc}\mathscr{V}^b_\mu J^{\mu c}_{\rm A} + f^{abc}\mathscr{A}^b_\mu J^{\mu c}_{\rm A}\Big)e^{i\int d^4x[i\overline{\psi}\gamma^\alpha(\partial_\alpha - i\mathscr{V}_\alpha - i\mathscr{A}_\alpha\gamma_5)\psi]}
$$

Expanding in perturbation theory, the terms with two gauge fields give the contribution of the triangle diagram. The **parity-violating** triangles ones are

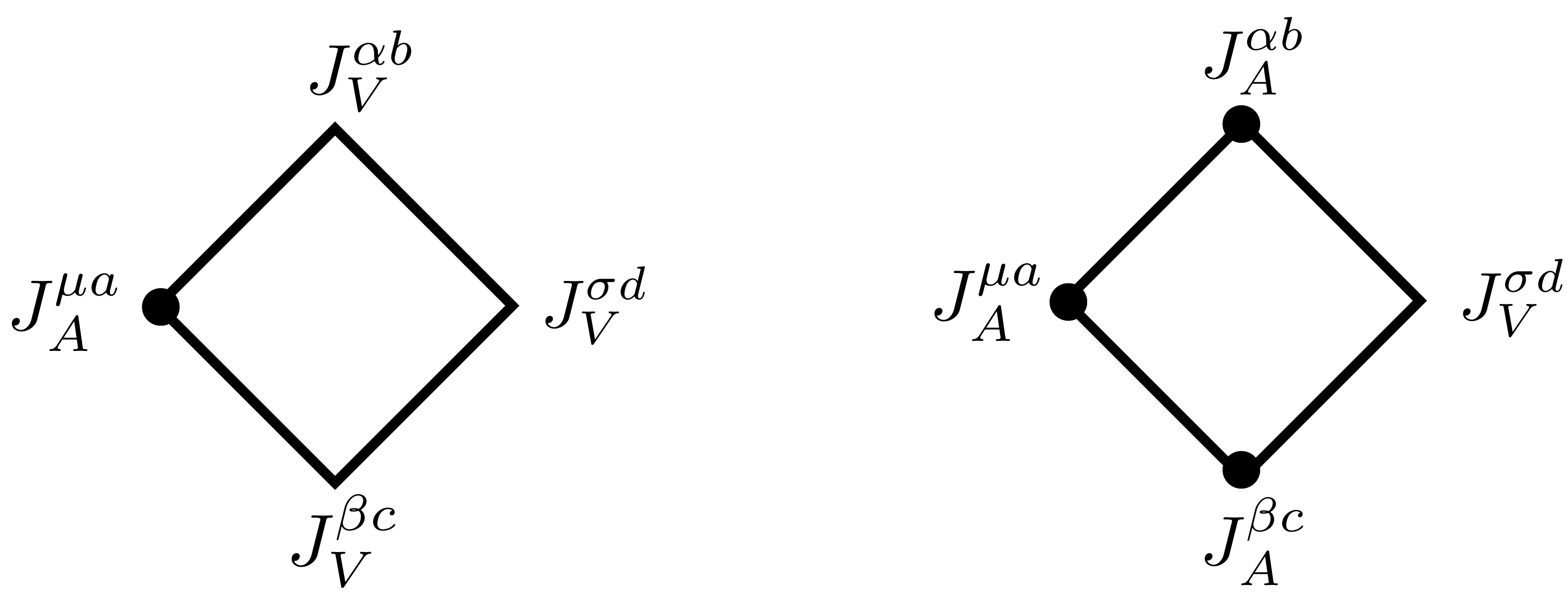




$$\text{Anomaly} = \langle(\partial_\mu J^{\mu a}_{\mathrm{A}} + f^{abc}\mathscr{V}^b_\mu J^{\mu c}_{\mathrm{A}} + f^{abc}\mathscr{A}^b_\mu J^{\mu c}_{\mathrm{A}})\rangle_{\mathscr{V},\mathscr{A}}$$

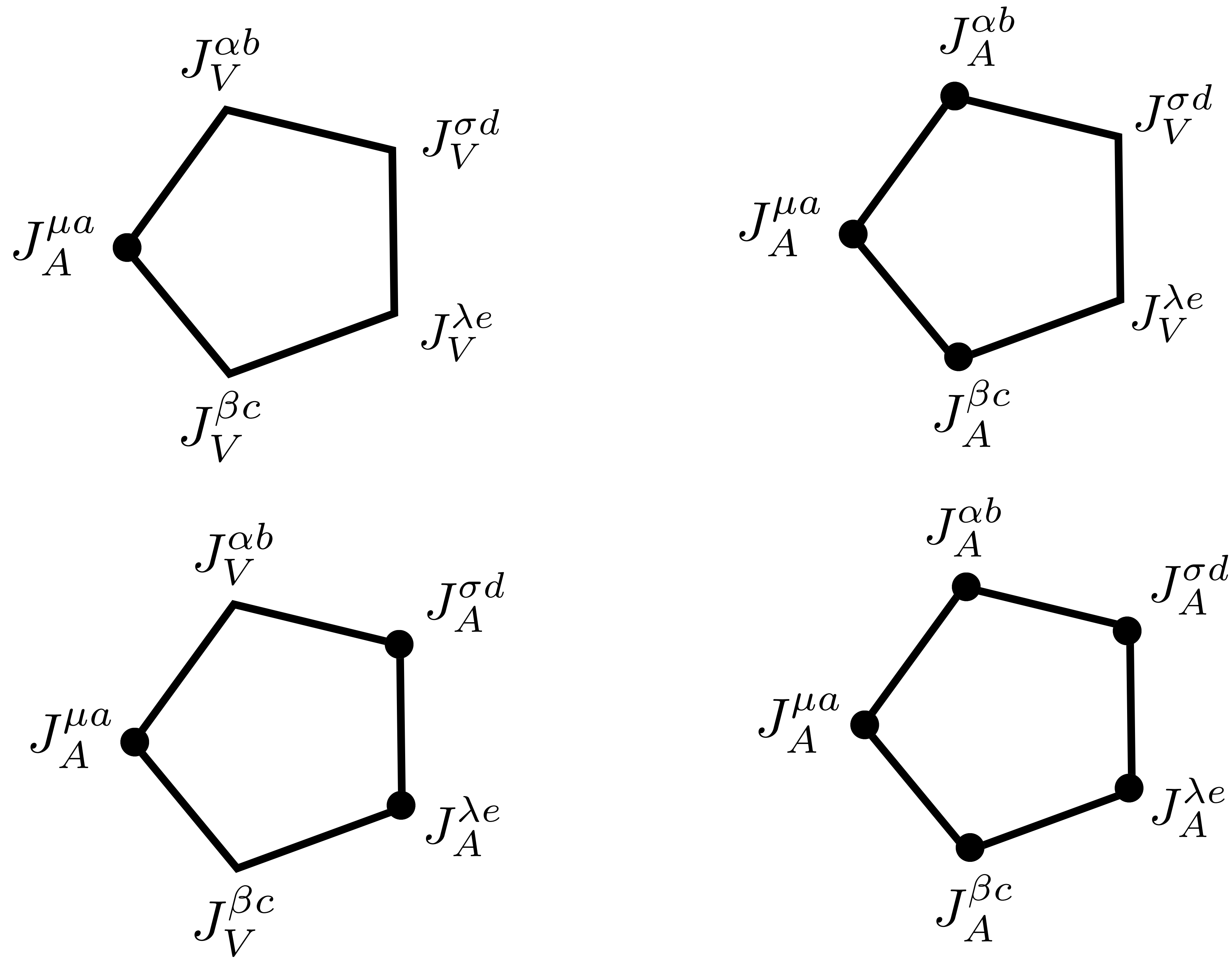
In the non-Abelian case, there are terms in the triangle with three gauge fields.

Their contribution **combines** with terms coming from the (logarithmically divergent) box diagrams



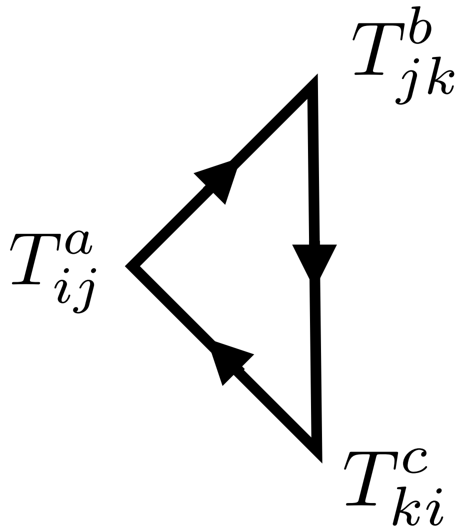
$$\text{Anomaly} = \langle (\partial_\mu J_{\rm A}^{\mu a} + f^{abc}\mathscr{V}_\mu^b J_{\rm A}^{\mu c} + f^{abc}\mathscr{A}_\mu^b J_{\rm A}^{\mu c})\rangle_{\mathscr{V},\mathscr{A}}$$

Finally, there are also contributions to the anomaly from the (UV finite) pentagon diagrams:



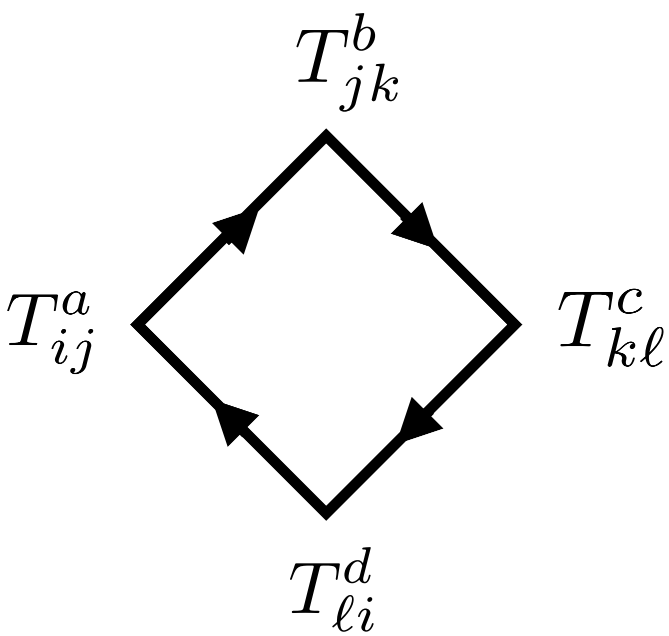
What about the group theory factors?

For **triangle** we have (AVV and AAA):



$$+\text{Bose symmetry} \longrightarrow \sim \mathrm{Tr}\left[T^a\{T^b,T^c\}\right]$$

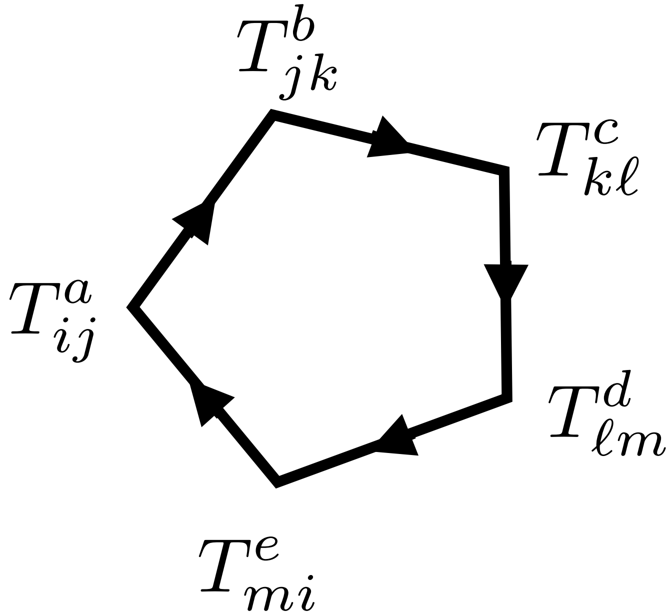
whereas the result for the **box** is (AVVV and AAAV):



$$+\text{Bose symmetry} \longrightarrow \sim \mathrm{Tr}\left[T^a\{T^b,[T^c,T^d]\}\right]$$

$$= \; if^{cde}\mathrm{Tr}\left[T^a\{T^b,T^e\}\right]$$

Finally, we deal with the **pentagon** (AVVVV, AVVAA, and AAAAA):



$$+\text{Bose symmetry} \qquad \sim \mathrm{Tr}\left[T^a T^{[b} T^c T^d T^{e]}\right]$$

$$\sim f^{r[bc} f^{de]s}\,\mathrm{Tr}\left[T^a \{T^r, T^s\}\right]$$

- The box and pentagon diagrams only contribute to non-Abelian case.
- The cancellation condition for the triangle diagram

$$\mathrm{Tr}\left[T^a \{T^b, T^c\}\right] = 0$$

automatically implies the **cancellation of the box and the pentagon** as well.

Therefore, to cancel the gauge anomaly we only have to care about the triangle!

Computing all these diagrams and imposing vector current conservation

$$\langle(\mathcal{D}_\mu J^\mu_{\rm V})^a\rangle_{\mathscr{V},\mathscr{A}} = 0$$

William A. Bardeen
(b. 1941)

one arrives at the expression of the **Bardeen anomaly**

$$\langle(\mathcal{D}_\mu J^\mu_{\rm A})^a\rangle_{\mathscr{V},\mathscr{A}} = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left\{T^a\left[\mathscr{V}_{\mu\nu}\mathscr{V}_{\alpha\beta}+\frac{1}{3}\mathscr{A}_{\mu\nu}\mathscr{A}_{\alpha\beta}\right.\right.$$

$$\left.\left.+\ \frac{8i}{3}\Big(\mathscr{A}_\mu\mathscr{A}_\nu\mathscr{V}_{\alpha\beta}+\mathscr{A}_\mu\mathscr{V}_{\nu\alpha}\mathscr{A}_\beta+\mathscr{V}_{\mu\nu}\mathscr{A}_\alpha\mathscr{A}_\beta\Big)-\frac{32}{3}\mathscr{A}_\mu\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right]\right\}$$

where

$$\mathscr{V}_{\mu\nu} = \partial_\mu\mathscr{V}_\nu-\partial_\nu\mathscr{V}_\mu-i[\mathscr{V}_\mu,\mathscr{V}_\nu]-i[\mathscr{A}_\mu,\mathscr{A}_\nu]$$

$$\mathscr{A}_{\mu\nu} = \partial_\mu\mathscr{A}_\nu-\partial_\nu\mathscr{A}_\mu-i[\mathscr{V}_\mu,\mathscr{A}_\nu]-i[\mathscr{A}_\mu,\mathscr{V}_\nu]$$

The result **preserve vector** gauge transformations (it depends on the vector field strength $\mathscr{V}_{\mu\nu}$ alone).

We can recast the **Bardeen** result for the case of a single **left- or right-handed fermion**

left: $\mathscr{V}_\mu = \mathscr{A}_\mu = \frac{1}{2}\mathscr{L}_\mu$ $\qquad J_L^\mu = \frac{1}{2}\Big(J_{\mathrm{V}}^\mu - J_{\mathrm{A}}^\mu\Big)$

right: $\mathscr{V}_\mu = -\mathscr{A}_\mu = \frac{1}{2}\mathscr{R}_\mu$ $\qquad J_R^\mu = \frac{1}{2}\Big(J_{\mathrm{V}}^\mu + J_{\mathrm{A}}^\mu\Big)$

For a **left-handed** fermion:

$$\langle(\mathcal{D}_\mu J_L^\mu)^a\rangle_{\mathscr{L}} = -\frac{1}{2}\langle(\mathcal{D}_\mu J_{\mathrm{A}}^\mu)^a\rangle_{\mathscr{V}=\mathscr{A}} = \frac{1}{96\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[T^a\Big(\mathscr{L}_{\mu\nu}\mathscr{L}_{\alpha\beta}\right.$$
$$\left. + i\mathscr{L}_\mu\mathscr{L}_\nu\mathscr{L}_{\alpha\beta} + i\mathscr{L}_\mu\mathscr{L}_{\nu\alpha}\mathscr{L}_\beta + i\mathscr{L}_{\mu\nu}\mathscr{L}_\alpha\mathscr{L}_\beta - 2\mathscr{L}_\mu\mathscr{L}_\nu\mathscr{L}_\alpha\mathscr{L}_\beta\Big)\right]$$

$$\langle(\mathcal{D}_\mu J_L^\mu)^a\rangle_{\mathscr{L}} = \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[T^a\partial_\mu\left(\mathscr{L}_\nu\partial_\alpha\mathscr{L}_\beta - \frac{i}{2}\mathscr{L}_\nu\mathscr{L}_\alpha\mathscr{L}_\beta\right)\right]$$

left: $\mathscr{V}_\mu = \mathscr{A}_\mu = \frac{1}{2}\mathscr{L}_\mu$ $\qquad J^\mu_L = \frac{1}{2}\Big(J^\mu_{\rm V} - J^\mu_{\rm A}\Big)$

right: $\mathscr{V}_\mu = -\mathscr{A}_\mu = \frac{1}{2}\mathscr{R}_\mu$ $\qquad J^\mu_R = \frac{1}{2}\Big(J^\mu_{\rm V} + J^\mu_{\rm A}\Big)$

and similarly for a **right-handed** fermion

$$\langle(\mathcal{D}_\mu J^\mu_R)^a\rangle_{\mathscr{R}} = \frac{1}{2}\langle(\mathcal{D}_\mu J^\mu_{\rm A})^a\rangle_{\mathscr{V}=-\mathscr{A}} = -\frac{1}{96\pi^2}\epsilon^{\mu\nu\alpha\beta}{\rm Tr}\Big[T^a\Big(\mathscr{R}_{\mu\nu}\mathscr{R}_{\alpha\beta} + i\mathscr{R}_\mu\mathscr{R}_\nu\mathscr{R}_{\alpha\beta} + i\mathscr{R}_\mu\mathscr{R}_{\nu\alpha}\mathscr{R}_\beta + i\mathscr{R}_{\mu\nu}\mathscr{R}_\alpha\mathscr{R}_\beta - 2\mathscr{R}_\mu\mathscr{R}_\nu\mathscr{R}_\alpha\mathscr{R}_\beta\Big)\Big]$$

$$\langle(\mathcal{D}_\mu J^\mu_R)^a\rangle_{\mathscr{R}} = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}{\rm Tr}\left[T^a\partial_\mu\left(\mathscr{R}_\nu\partial_\alpha\mathscr{R}_\beta - \frac{i}{2}\mathscr{R}_\nu\mathscr{R}_\alpha\mathscr{R}_\beta\right)\right]$$

opposite sign!

We have seen that the condition for the **cancellation** of the **gauge non-Abelian anomaly** reads

$$\mathrm{Tr}\left[T^a\left\{T^b,T^c\right\}\right]=0$$

In a theory with N_+ positive chirality fermions and N_- negative chirality fermions, the **anomaly cancellation condition** takes the form

$$\boxed{\sum_{i=1}^{N_+}\mathrm{Tr}\left[T^a_{i,+}\left\{T^b_{i,+},T^c_{i,+}\right\}\right]-\sum_{i=1}^{N_-}\mathrm{Tr}\left[T^a_{i,-}\left\{T^b_{i,-},T^c_{i,-}\right\}\right]=0}$$

Are there **"safe" representations** for which

$$d^{abc}_{\mathbf{R}}\equiv\mathrm{Tr}\left[T^a_{\mathbf{R}}\left\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\right\}\right]=0 \qquad \textbf{(anomaly coefficients)}$$

Let us do some **group theory**...

A Lie algebra representation is **real** or **pseudoreal** if there is an **intertwining operator** S relating the **representation** and its **complex conjugate**

$$T_{\mathbf{R}}^{a\,*} = -S T_{\mathbf{R}}^a S^{-1} \qquad \begin{cases} S^T = S & \text{real} \\ S^T = -S & \text{pseudoreal} \end{cases}$$

Using

$$\mathrm{Tr}\left[T_{\mathbf{R}}^a\left\{T_{\mathbf{R}}^b, T_{\mathbf{R}}^c\right\}\right] = \mathrm{Tr}\left[T_{\mathbf{R}}^a\left\{T_{\mathbf{R}}^b, T_{\mathbf{R}}^c\right\}\right]^T = \mathrm{Tr}\left[(T_{\mathbf{R}}^a)^*\left\{(T_{\mathbf{R}}^b)^*, (T_{\mathbf{R}}^c)^*\right\}\right]$$

we find for **real** and **pseudoreal** representations

$$\mathrm{Tr}\left[(T_{\mathbf{R}}^a)^*\left\{(T_{\mathbf{R}}^b)^*, (T_{\mathbf{R}}^c)^*\right\}\right] = -\mathrm{Tr}\left[S T_{\mathbf{R}}^a S^{-1}\left\{S T_{\mathbf{R}}^b S^{-1}, S T_{\mathbf{R}}^c S^{-1}\right\}\right] = -\mathrm{Tr}\left[T_{\mathbf{R}}^a\left\{T_{\mathbf{R}}^b, T_{\mathbf{R}}^c\right\}\right]$$

$$\Downarrow$$

$$\mathrm{Tr}\left[T_{\mathbf{R}}^a\left\{T_{\mathbf{R}}^b, T_{\mathbf{R}}^c\right\}\right] = -\mathrm{Tr}\left[T_{\mathbf{R}}^a\left\{T_{\mathbf{R}}^b, T_{\mathbf{R}}^c\right\}\right] \quad \Longrightarrow \quad d_{\mathbf{R}}^{abc} = 0$$

Thus, **real and pseudoreal** are **anomaly-free** representations

$$d_{\mathbf{R}}^{abc} = \mathrm{Tr}\left[T_{\mathbf{R}}^{a}\left\{T_{\mathbf{R}}^{b}, T_{\mathbf{R}}^{c}\right\}\right] = 0 \qquad \text{for } \mathbf{R} \text{ real or pseudoreal}$$

All representations of the following groups are **safe**

- SU(2)
- SO(2N+1)
- SO(4N) for $N \geq 2$
- Sp(2N) for $N \geq 3$
- and the exceptional groups $G_2,\ F_4,\ E_7,\ E_8$

Other groups whose representations are **neither real or pseudoreal** but are still **safe** are

- SO(4N+2) for $N \geq 2$
- E_6

In addition, the **adjoint** representation of any group is real and therefore **safe**.

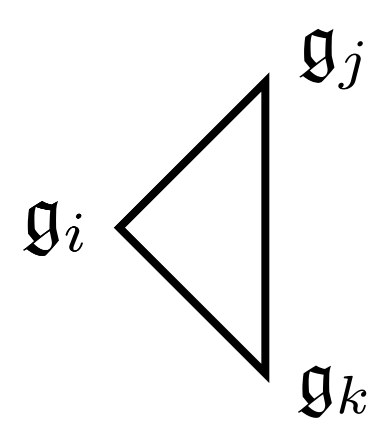
Potentially dangerous Lie group are

- U(1).
- SU(N) for $N \geq 3$.

For **non-safe groups**, anomalies can be **eliminated** either by choosing an **anomaly free representation** or **by cancellation**

$$\sum_{i=1}^{N_+} \mathrm{Tr}\left[T^a_{i,+}\left\{T^b_{i,+}, T^c_{i,+}\right\}\right] - \sum_{i=1}^{N_-} \mathrm{Tr}\left[T^a_{i,-}\left\{T^b_{i,-}, T^c_{i,-}\right\}\right] = 0$$

If the gauge group is a direct product, $G_1 \otimes ... \otimes G_n$, there might be **mixed gauge anomalies** associated with triangles with "different group factors" at each vertex



Gravitational anomalies

Gravitons are quantized perturbations over flat (or any other background) spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \qquad\qquad (\ \kappa = \sqrt{8\pi G_N} \ )$$

The graviton action is obtained expanding the **Einstein-Hilbert action** around the Minkowski metric

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R[g]$$

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \frac{1}{2}\partial^\alpha h_{\alpha\beta}\partial_\mu h^{\mu\beta} + \text{self-interactions}\right)$$

At the level of the graviton field, diffeomorphism invariance translate into gauge tranformations generated by a vector field

$$\delta h_{\mu\nu}(x) = \frac{1}{2}\Big[\partial_\mu\xi_\nu(x) + \partial_\nu\xi_\mu(x)\Big]$$

Expanding the matter action to **linear order** in the graviton field

$$S[\phi_i,\eta+2\kappa h] = S[\phi_i] + 2\kappa\left(\int d^4x\, h_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}\right)\Bigg|_{g=\eta}$$

$$= S[\phi_i] - \kappa\left[\int d^4x\sqrt{-g}h_{\mu\nu}\left(-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\right)\right]\Bigg|_{g=\eta}$$

leads to the **coupling** between the graviton and the energy-momentum tensor

$$S_{\rm int} = -\kappa\int d^4x\, h_{\mu\nu}T^{\mu\nu}$$

Invariance under gauge transformations

$$\delta h_{\mu\nu}(x) = \frac{1}{2}\Big[\partial_\mu\xi_\nu(x)+\partial_\nu\xi_\mu(x)\Big]$$

depends on the **conservation of the energy-momentum tensor**

$$\delta S_{\rm int} = \kappa\int d^4x\,\xi_\nu\partial_\mu T^{\mu\nu} \quad\Longrightarrow\quad \partial_\mu T^{\mu\nu}=0$$

Gravitational anomalies appear whenever the **energy-momentum tensor is not conserved quantum-mechanically**

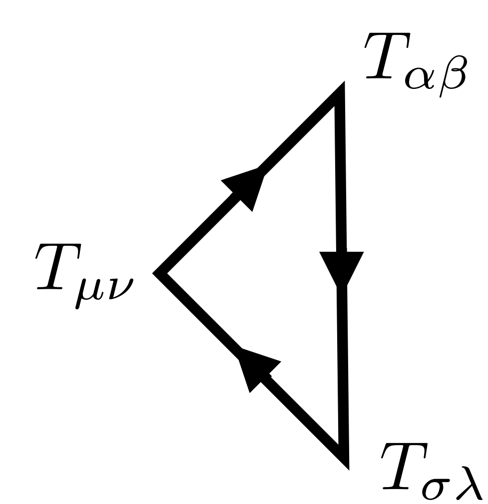
$$\partial_\mu \langle T^{\mu\nu}(x)\rangle_h \neq 0$$

Let us consider a theory of a chiral fermion coupled to a background graviton field

$$T_{\mu\nu} = \frac{i}{4}\overline{\psi}\Big(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu\Big)\psi \qquad \text{where} \qquad f_1 \overleftrightarrow{\partial}_\nu f_2 = f_1(\partial_\mu f_2) - (\partial_\mu f_1) f_2$$

The expectation value of the energy-momentum tensor is then

$$\langle T^{\mu\nu}(x)\rangle_h = \frac{1}{Z}\int \mathscr{D}\psi \mathscr{D}\overline{\psi}\, T^{\mu\nu}(x)\, e^{i\int d^4y(i\overline{\psi}_+\gamma^\mu\partial_\mu\psi_+ - \kappa h^{\mu\nu}T_{\mu\nu})}$$



Expanding in powers of κ we find again the **triangle diagram**, this time with three energy-momentum tensor insertions

But, since anomalies and parity noninvariance come together, the question is whether **gravitational couplings** are **sensitive to chirality**

This depends on the dimension:

- $D = 4k$:

CPT **reverses** the helicity of fermions

- $D = 4k+2$:

CPT **preserves** the helicity of fermions

Thus, in $D = 4k$ there are as many left-handed as right-handed fermions

+ "equivalence principle"

Gravitational couplings are chirality-blind

There are no pure gravitational anomalies in four dimensions

However, gravity can **contribute** to the **gauge anomaly**…

For example, a left-handed fermion coupled to a gauge field also couples to gravity through

$$S = \int d^4x \left[i\overline{\psi}\gamma^\mu \partial_\mu \psi + \overline{\psi}\gamma^\mu T^a \left(\frac{1-\gamma_5}{2} \right) \psi \mathscr{A}^a_\mu - \kappa h_{\mu\nu} T^{\mu\nu} \right]$$

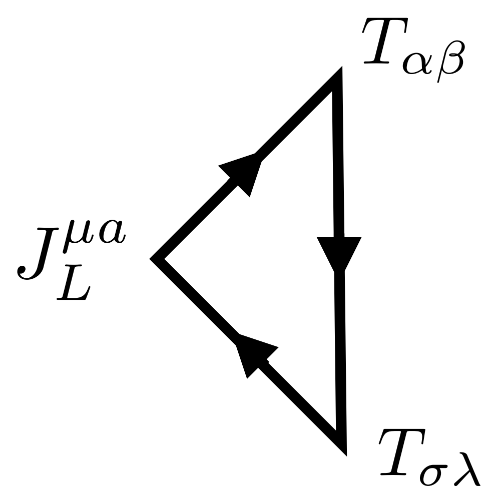
The quantum conservation of the current is then

$$\langle J^{\mu a}_L(x) \rangle_{\mathscr{A},h} = \frac{1}{Z} \int \mathscr{D}\psi \mathscr{D}\overline{\psi} J^{\mu a}_L(x) e^{iS_0[\psi,\overline{\psi},\mathscr{A}] - i\kappa \int d^4y\, T^{\mu\nu} h_{\mu\nu}}$$

Expanding in powers of κ brings down insertions of the energy-momentum tensor into the correlation function. Then, we have contributions like

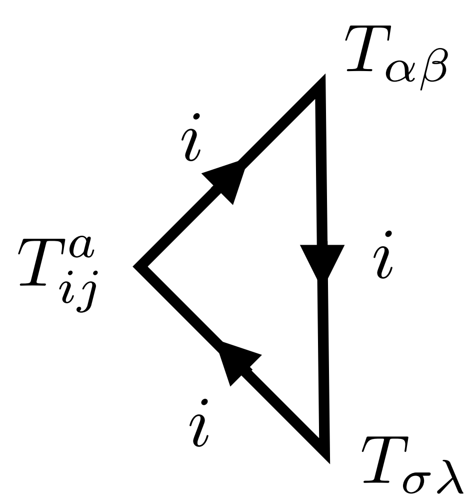
$$-\frac{\kappa^2}{2} \int d^4y_1 \int d^4y_2 \, \langle 0 | T \left[J^{\mu a}_L(x) T^{\alpha\beta}(y_1) T^{\sigma\lambda}(y_2) \right] | 0 \rangle h_{\alpha\beta}(y_1) h_{\sigma\lambda}(y_2)$$

Diagrammatically, we have again a **triangle** diagram with **one gauge** current and **two energy-momentum tensors**



$+$Bose symmetry

Since we are only interested in cancelling this contribution we just need to look at the group theory factor



$\sim \; \mathrm{Tr}\, T^a$

Thus, the condition for the cancellation of **mixed gauge-gravitational anomalies** is

$$\sum_{\text{right-handed}} \mathrm{Tr}\, T_{+}^{a} - \sum_{\text{left-handed}} \mathrm{Tr}\, T_{-}^{a} = 0$$

- $\mathrm{SU}(N)$ for $N \geq 2$ do not contribute to mixed anomalies (tracelessness!)

- But **beware** of **U(1)**'s!!!

The cancellation of mixed anomalies poses very **strong nontrivial constraint** on theories (e.g. the standard model, MSSM,…).

Functional methods

Foreword: Euclidean fermion fields

In Minkowski space, the Dirac matrices satisfy $[\,\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)]$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \quad \Longrightarrow \quad \left\{ \begin{array}{l} \gamma^{0\dagger} = \gamma^0 \\ \gamma^{i\dagger} = -\gamma^i \end{array} \right.$$

Dirac fermions are defined as objects transforming under the Lorentz group as

$$\psi' = e^{-\frac{i}{2}\vartheta_{\mu\nu}\sigma^{\mu\nu}}\psi \equiv U(\vartheta)\psi \qquad \text{where} \qquad \left\{ \begin{array}{ll} \sigma^{\mu\nu} = -\dfrac{i}{4}[\gamma^\mu,\gamma^\nu] & \\ \sigma^{0i\dagger} = -\sigma^{0i}, & \sigma^{ij\dagger} = \sigma^{ij}. \end{array} \right.$$

Since $\sigma^{\mu\nu}$ is not Hermitian, Hermitian conjugate spinors are not "contravariant"

$$\psi^{\dagger\prime} = \psi^\dagger e^{\frac{i}{2}\vartheta_{\mu\nu}\sigma^{\mu\nu\dagger}} \equiv \psi^\dagger U(\vartheta)^\dagger \neq \psi^\dagger U(\vartheta)^{-1}$$

$$\sigma^{\mu\nu\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0 \quad \Downarrow \quad \gamma^0 U(\vartheta)^\dagger\gamma^0 = U(\vartheta)^{-1}$$

$$\overline{\psi}' = \psi^{\dagger\prime}\gamma^0 = \psi^\dagger U(\vartheta)^\dagger\gamma^0 = \psi^\dagger\gamma^0 U(\vartheta)^{-1} = \overline{\psi}U(\vartheta)^{-1}$$

Euclidean space can be obtained by Wick rotation from Minkowski signature

$$x^0 = -ix^4 \quad \Longrightarrow \quad \eta_{\mu\nu} \longrightarrow -\delta_{\mu\nu}$$

while the new Dirac matrices are defined as

$$\left.\begin{aligned} \widehat{\gamma}^4 &= i\gamma^0 \\ \widehat{\gamma}^i &= \gamma^i \end{aligned}\right\} \quad \Longrightarrow \quad \left\{\begin{aligned} &\{\widehat{\gamma}^\mu, \widehat{\gamma}^\nu\} = -2\delta^{\mu\nu}\mathbb{I} \\ &\widehat{\gamma}^{\mu\dagger} = -\widehat{\gamma}^\mu \end{aligned}\right.$$

Euclidean Dirac fermions are objects transforming under $\mathrm{SO}(4)$ as

$$\psi' = e^{-\frac{i}{2}\omega_{\mu\nu}\widehat{\sigma}^{\mu\nu}}\psi \equiv O(\omega)\psi \quad \Longrightarrow \quad \left\{\begin{aligned} &\widehat{\sigma}^{\mu\nu} = \frac{i}{4}[\widehat{\gamma}^\mu, \widehat{\gamma}^\nu] \\ &\widehat{\sigma}^{\mu\nu\dagger} = \widehat{\sigma}^{\mu\nu} \end{aligned}\right.$$

Now, Hermitian conjugate objects are **contravariant**

$$\psi'^\dagger = \psi^\dagger e^{\frac{i}{2}\omega_{\mu\nu}\widehat{\sigma}^{\mu\nu\dagger}} \equiv \psi^\dagger O(\omega)^\dagger = \psi^\dagger O(\omega)^{-1}$$

In Euclidean QFT, ψ and $\psi^\dagger$ are considered **independent variables**.

In Euclidean space, the chirality matrix is defined as

$$\widehat{\gamma}_5 = -\widehat{\gamma}^1\widehat{\gamma}^2\widehat{\gamma}^3\widehat{\gamma}^4$$

satisfying

$$\widehat{\gamma}_5^\dagger = \widehat{\gamma}_5$$

A particularly important identity in the computation of anomalies is

$$\mathrm{Tr}\left(\widehat{\gamma}_5\widehat{\gamma}^\mu\widehat{\gamma}^\nu\widehat{\gamma}^\alpha\widehat{\gamma}^\beta\right) = -4\epsilon^{\mu\nu\alpha\beta} \qquad \text{where} \qquad \epsilon^{1234} = 1$$

Comparing with its Minkowskian counterpart

$$\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = -4i\epsilon^{\mu\nu\alpha\beta} \qquad \text{with} \qquad \epsilon^{0123} = 1$$

we see how Euclidean chiral anomalies will have an **addition** factor of i.

Notation **WARNING**

From now on, Euclidean gamma matrices will be "**hatless**"

We denote $\psi(x)^\dagger \equiv \overline{\psi}(x) \neq \psi(x)^\dagger \widehat{\gamma}^0$

The fermion (Euclidean) effective action

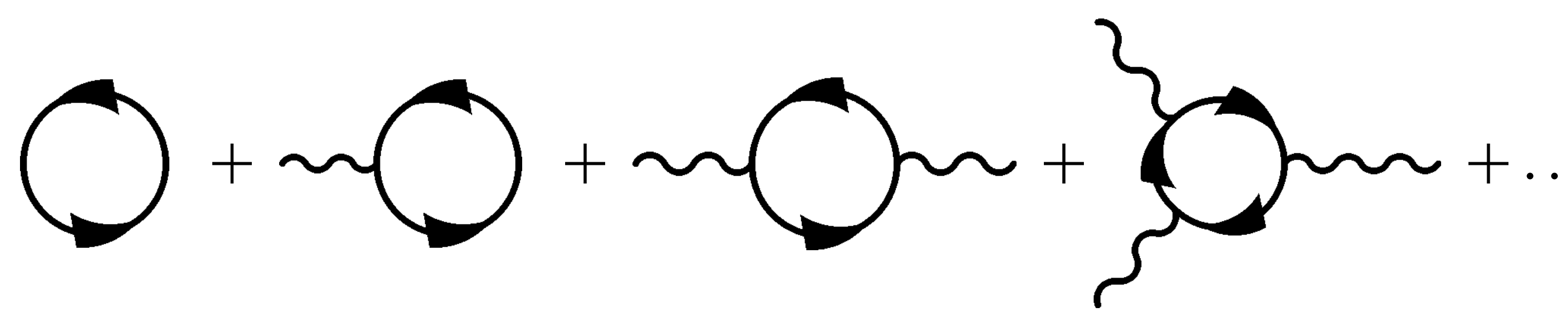
As above, we study a **massless fermion** coupled to an **external gauge field** $\mathscr{A}_\mu = \mathscr{A}^a T^a$ and define the Euclidean **fermion effective action** in ***d*** **dimensions**.

$$e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[-\int d^d x \, \overline{\psi}\gamma^\mu\Big(i\partial_\mu + \mathscr{A}_\mu\Big)\psi\right]$$

which is **nonlocal**, since we are **integrating out a massless state**.

Expanding the action in powers of the **external gauge field**, we see that this sums the contribution of all **one-loop diagrams** with arbitrary gauge field insertions:

$\Gamma[\mathscr{A}] =$



$$e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, \exp\left[-\int d^dx\,\overline{\psi}\gamma^\mu\Big(i\partial_\mu + \mathscr{A}_\mu\Big)\psi\right] \qquad\qquad \mathscr{A}_\mu = \mathscr{A}^a T^a$$

Let us carry out a **gauge transformation** on the external field:

$$\delta_u \mathscr{A}^a_\mu = \partial_\mu u^a + f^{abc}\mathscr{A}^b_\mu u^c \equiv (\mathcal{D}_\mu u)^a$$

The corresponding **transformation of the effective action** is given by

$$\delta_u\Gamma[\mathscr{A}] = \int d^dx\,\delta_u\mathscr{A}^a_\mu(x)\frac{\delta\Gamma[\mathscr{A}]}{\delta\mathscr{A}^a_\mu(x)}$$

On the other hand

$$-\frac{\delta\Gamma[\mathscr{A}]}{\delta\mathscr{A}^a_\mu(x)}e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi\Big[-\overline{\psi}(x)\gamma^\mu T^a\psi(x)\Big]\exp\left[-\int d^dy\,\overline{\psi}\gamma^\alpha\Big(i\partial_\alpha + \mathscr{A}_\alpha\Big)\psi\right]$$

$$-\frac{\delta\Gamma[\mathscr{A}]}{\delta\mathscr{A}_\mu^a(x)}e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi\Big[-\overline{\psi}(x)\gamma^\mu T^a\psi(x)\Big]\exp\left[-\int d^dy\,\overline{\psi}\gamma^\alpha\Big(i\partial_\alpha + \mathscr{A}_\alpha\Big)\psi\right]$$

$$\frac{\delta\Gamma[\mathscr{A}]}{\delta\mathscr{A}_\mu^a(x)} = \frac{1}{e^{-\Gamma[\mathscr{A}]}}\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\Big[\overline{\psi}(x)\gamma^\mu T^a\psi(x)\Big]\exp\left[-\int d^dy\,\overline{\psi}\gamma^\alpha\Big(i\partial_\alpha + \mathscr{A}_\alpha\Big)\psi\right]$$

Z

$J^{\mu a}(x)$

We identify the **expectation value** of the **gauge current**

$$\frac{\delta\Gamma[\mathscr{A}]}{\delta\mathscr{A}_\mu^a(x)} = \langle\overline{\psi}(x)\gamma^\mu T^a\psi(x)\rangle_{\mathscr{A}} = \langle J^{\mu a}(x)\rangle_{\mathscr{A}}$$

$$\delta_u\Gamma[\mathscr{A}] = \int d^dx\,\delta_u\mathscr{A}_\mu^a(x)\frac{\delta\Gamma[\mathscr{A}]}{\delta\mathscr{A}_\mu^a(x)}$$

$$\delta_u\Gamma[\mathscr{A}] = \int d^dx\,\delta_u\mathscr{A}_\mu^a(x)\langle J^{\mu a}(x)\rangle_{\mathscr{A}}$$

$$\delta_u \mathscr{A}_\mu^a = (\mathcal{D}_\mu u)^a \qquad\qquad \delta_u\Gamma[\mathscr{A}] = \int d^dx\, \delta_u\mathscr{A}_\mu^a(x)\langle J^{\mu a}(x)\rangle_{\mathscr{A}}$$

Thus, the variation of the fermion effective action is given by

$$\delta_u\Gamma[\mathscr{A}] = \int d^dx\, (\mathcal{D}_\mu u)^a(x)\langle J^{\mu a}(x)\rangle_{\mathscr{A}}$$

and integrating by parts

$$\delta_u\Gamma[\mathscr{A}] = -\int d^dx\, u^a(x)[\mathcal{D}_\mu\langle J^\mu(x)\rangle_{\mathscr{A}}]_a$$

Identifying the **potential gauge anomaly**, we arrive at

$$[\mathcal{D}_\mu\langle J^\mu(x)\rangle_{\mathscr{A}}]_a \equiv \mathscr{G}_a[\mathscr{A}(x)] \quad \longrightarrow \quad \delta_u\Gamma[\mathscr{A}] = -\int d^dx\, u^a(x)\mathscr{G}_a[\mathscr{A}(x)]$$

$$\delta_u \Gamma[\mathscr{A}] = -\int d^d x\, u^a(x) \mathscr{G}_a[\mathscr{A}(x)]$$

We conclude that the **gauge variation** of the fermion **effective action** is determined by the **anomaly**.

In fact, we can write the **anomaly** as

$$\boxed{\mathscr{G}_a[\mathscr{A}(x)] = -\frac{\delta}{\delta u^a(x)} \Gamma[\mathscr{A} + \mathcal{D}u]\Big|_{u=0}}$$

Thus, one way to **compute** the **anomaly** is by directly **constructing** the fermion **effective action** for the corresponding gauge theory

Differential geometry

But, how do the **anomaly transform** under **gauge** transformations?

Remember that under **finite gauge transformations**

$$\mathscr{A}_\mu^g = g^{-1}\mathscr{A}_\mu g + i g^{-1}\partial_\mu g \qquad \xrightarrow{g \approx 1 - iu^a T_a} \qquad \delta_u \mathscr{A}_\mu = \mathcal{D}_\mu u$$

$$\mathscr{F}_\mu^g = g^{-1}\mathscr{F}_{\mu\nu} g \qquad\qquad \delta_u \mathscr{F}_{\mu\nu} = -i[\mathscr{F}_{\mu\nu}, u]$$

Composing two infinitesimal transformations $g_1 \approx 1 - iu, \quad g_2 \approx 1 - iv$

$$\mathscr{A}_\mu^{g_1 g_2} - \mathscr{A}_\mu^{g_2 g_1} = (\delta_u\delta_v - \delta_v\delta_u)\mathscr{A}_\mu = \mathcal{D}_\mu[u,v]$$

$$\mathscr{F}_{\mu\nu}^{g_1 g_2} - \mathscr{F}_{\mu\nu}^{g_2 g_1} = (\delta_u\delta_v - \delta_v\delta_u)\mathscr{F}_{\mu\nu} = -i\big[\mathscr{F}_{\mu\nu}, [u,v]\big]$$

In general, acting on any quantity:

$$\delta_u\delta_v - \delta_v\delta_u = \delta_{[u,v]}$$

$$\delta_u\delta_v - \delta_v\delta_u = \delta_{[u,v]}$$

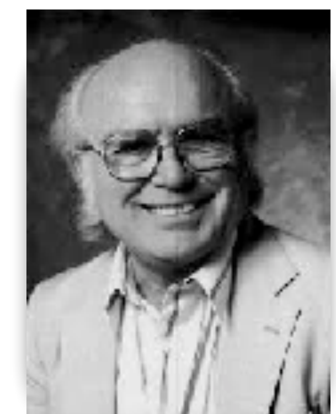
Julius Wess
(1934-2007)

Applying it to the fermion **effective action**

$$(\delta_u\delta_v - \delta_v\delta_u)\Gamma[\mathscr{A}] = \delta_{[u,v]}\Gamma[\mathscr{A}]$$

$$\Downarrow$$

$$\delta_u\Big(\delta_v\Gamma[\mathscr{A}]\Big) - \delta_v\Big(\delta_u\Gamma[\mathscr{A}]\Big) = \delta_{[u,v]}\Gamma[\mathscr{A}]$$



Bruno Zumino
(1923-2014)

But now, if we recall that

$$\delta_u\Gamma[\mathscr{A}] = -\int d^dx\, u^a\mathscr{G}_a[\mathscr{A}]$$

we arrive at the **Wess-Zumino consistency conditions**

$$\boxed{\int d^dx\, v^a\delta_u\mathscr{G}_a[\mathscr{A}] - \int d^dx\, u^a\delta_v\mathscr{G}_a[\mathscr{A}] = \int d^dx\, [u,v]^a\mathscr{G}_a[\mathscr{A}]}$$

$$\int d^dx\, v^a \delta_u \mathscr{G}_a[\mathscr{A}] - \int d^dx\, u^a \delta_v \mathscr{G}_a[\mathscr{A}] = \int d^dx\, [u,v]^a \mathscr{G}_a[\mathscr{A}]$$

- **Any anomaly** derived as the **variation** of a **functional** automatically **satisfies** the Wess-Zumino **consistency condition**.

- If the **functional** is local, there is **no anomaly**, since the gauge variation of the functional can be **cancelled** by adding a **local counterterm**.

- **Only solutions** to the Wess-Zumino consistency conditions derived from **nonlocal functionals** can be considered to **candidates** to represent **anomalies**.

To find these **nontrivial solutions** to the Wess-Zumino equations we need a bit of **differential geometry**.

A short detour into mathematics

Differential forms: definitions and conventions (Nakahara, 2nd edition 2003)

Differential forms are elements of the algebra of **totally antisymmetric covariant tensors** of rank $p \leq d$. They are spanned by

$$dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \sum_{\sigma \in S_p} (-1)^{\pi(\sigma)} dx^{\mu_{\sigma(1)}} \otimes \ldots \otimes dx^{\mu_{\sigma(p)}}$$

- **p-forms:** $\omega_p \in \Omega_p^d(\mathcal{M}), \;\; 0 \leq p \leq d$

no wedges!

$$\omega_p = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \equiv \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \ldots dx^{\mu_p}$$

- **exterior product:** $\wedge : \Omega_p^d(\mathcal{M}) \otimes \Omega_q^d(\mathcal{M}) \longrightarrow \Omega_{p+q}^r(\mathcal{M})$

$$\omega_p \eta_q = \frac{1}{p!q!} \omega_{\mu_1 \ldots \mu_p} \eta_{\nu_1 \ldots \nu_q} dx^{\mu_1} \ldots dx^{\mu_p} dx^{\nu_1} \ldots dx^{\nu_q}$$

$$\omega_p \eta_q = (-1)^{pq} \eta_q \omega_q \quad \Longrightarrow \quad \omega_p^2 = 0 \quad \text{for } p \text{ odd}$$

Differential forms: definitions and conventions (Nakahara, 2nd edition 2003)

- **exterior differential:** $d:\Omega_p^d(\mathcal{M})\longrightarrow\Omega_{p+1}^r(\mathcal{M})$

$$d\omega_p=\frac{1}{p!}\partial_\alpha\omega_{\mu_1}\ldots\omega_{\mu_p}dx^\alpha dx^{\mu_1}\ldots dx^{\mu_p}$$

$$d(\omega_p\eta_q)=(d\omega_p)\eta_q+(-1)^p\omega_p(d\eta_q)$$ (Leibniz rule)

$$d^2\omega_p=0$$ (nihilpotency)

$$d\omega_d=0$$

There are two important **definitions**:

- **Closed p-form**:

$$d\omega_p=0$$

- **Exact p-form**: there is $\eta_{p-1}\in\Omega_p^d(\mathcal{M})$ such that

$$\omega_p=d\eta_{p-1}$$

Differential forms: definitions and conventions (Nakahara, 2nd edition 2003)

All exact forms are closed but, what about the converse?

Henri Poincaré
(1854-1912)

Poincaré lemma:

"All closed forms are **locally** exact"

$$d\omega_p = 0 \quad \xrightarrow{\text{locally}} \quad \omega_p = d\eta_{p-1}$$

Globally, this is **not necessarily true**.

- **Integration of differential forms:** a p-form $\omega_p \in \Omega_p^d(\mathcal{M})$ can be integrated over a p-dimensional open set $C_p \subset \mathcal{M}$

$$I = \int_{C_p} \omega_p$$

Differential forms: definitions and conventions (Nakahara, 2nd edition 2003)

- **Hodge dual:** $\star : \Omega_p^d(\mathcal{M}) \longrightarrow \Omega_{d-p}^d(\mathcal{M})$

$$\star\omega_p = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} \star (dx^{\mu_1} \ldots dx^{\mu_p})$$

$$\boxed{\begin{aligned} \epsilon_{01\ldots d-1} &= 1 \\ \epsilon^{01\ldots d-1} &= g^{-1} \end{aligned}}$$

with

$$\star(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}) = \frac{\sqrt{|g|}}{(d-r)!}\epsilon^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{d-r}} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{d-r}}$$

The Hodge dual is **only defined** in **spaces with a metric**.

$$\int_{\mathcal{M}} \omega_p \eta_{d-p} \longrightarrow \text{metric independent (topological)}$$

$$\int_{\mathcal{M}} \omega_p (\star\omega_p) \longrightarrow \text{depends on the metric}$$

Differential forms: definitions and conventions (Nakahara, 2nd edition 2003)

- **Hodge dual:** $\star : \Omega^d_p(\mathcal{M}) \longrightarrow \Omega^d_{d-p}(\mathcal{M})$

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metric!

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The Hodge dual is **only defined** in **spaces with a metric**.

$\int_{\mathcal{M}}\omega_p\eta_{d-p}$ ➡ metric independent (topological)

$\int_{\mathcal{M}}\omega_p(\star\omega_p)$ ➡ depends on the metric

Differential forms: definitions and conventions (Nakahara, 2nd edition 2003)

- **Stokes theorem:** $\omega_p \in \Omega^d_p(\mathcal{M})$ a p-form and $C_{p+1} \subset \mathcal{M}$ an open set with boundary ∂C_{p+1}

$$\int_{C_{p+1}} d\omega_p = \int_{\partial C_{p+1}} \omega_p$$

Gauge theory in the language of **differential forms**

Associated with the **gauge potential**, we construct the **gauge-algebra-valued one-form**

$$\mathcal{A} = -i\mathscr{A}_\mu dx^\mu = -i\mathscr{A}_\mu^a T^a dx^\mu$$

which is by construction antihermitian $\mathcal{A}^\dagger = -\mathcal{A}$

Associated with it, we construct the **field strength two-form**

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$$

which in components read

$$\mathcal{F} = -i\partial_\mu \mathscr{A}_\nu dx^\mu dx^\nu - \mathscr{A}_\mu \mathscr{A}_\nu dx^\mu dx^\nu = -i\Big(\partial_\mu \mathscr{A}_\nu - i\mathscr{A}_\mu \mathscr{A}_\nu\Big) dx^\mu dx^\nu$$

$$= -\frac{i}{2}\Big(\partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu - i[\mathscr{A}_\mu, \mathscr{A}_\nu]\Big) dx^\mu dx^\nu \equiv -\frac{i}{2}\mathscr{F}_{\mu\nu} dx^\mu dx^\nu$$

Gauge theory in the language of **differential forms**

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$$= -\frac{i}{2}\Big(\partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu - i[\mathscr{A}_\mu,\mathscr{A}_\nu]\Big)dx^\mu dx^\nu \equiv -\frac{i}{2}\mathscr{F}_{\mu\nu}dx^\mu dx^\nu$$

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$$

Taking the exterior differential on the field strength, we have

$$d\mathcal{F} = d^2\mathcal{A} + d\mathcal{A}\mathcal{A} - \mathcal{A}d\mathcal{A} = d\mathcal{A}\mathcal{A} - \mathcal{A}d\mathcal{A}$$

Using now $d\mathcal{A} = \mathcal{F} - \mathcal{A}^2$

$$d\mathcal{F} = \Big(\mathcal{F} - \mathcal{A}^2\Big)\mathcal{A} - \mathcal{A}\Big(\mathcal{F} - \mathcal{A}^2\Big) = \mathcal{F}\mathcal{A} - \mathcal{A}\mathcal{F} - \mathcal{A}^3 + \mathcal{A}^3$$

and we arrive at the **Bianchi identity**

$$d\mathcal{F} = \mathcal{F}\mathcal{A} - \mathcal{A}\mathcal{F}$$

In the **Abelian case**, $\mathcal{F}$ and $\mathcal{A}$ **commute** and we have

$$d\mathcal{F} = 0 \qquad \text{(remember that } \mathcal{F} = d\mathcal{A}\text{)}$$

We implement now **gauge transformations**

$$g = e^{-iu^a T^a} \equiv e^u$$

The transformation of the **connection one-form** is given by

$$\mathcal{A}_g = g^{-1}\mathcal{A}g + g^{-1}dg \qquad \text{(Abelian: } \mathcal{A}_g = \mathcal{A} + g^{-1}dg\text{)}$$

while the **field strength two-form** transform as an **adjoint** field

$$\mathcal{F}_g = g^{-1}\mathcal{F}g \qquad \text{(Abelian: } \mathcal{F}_g = \mathcal{F}\text{)}$$

For **infinitesimal** gauge transformations $g = 1 + u$ and $g^{-1} = 1 - u$

$$\delta_u \mathcal{A} = du + [\mathcal{A}, u] \qquad \text{(Abelian: } \delta_u \mathcal{A} = du\text{)}$$

and

$$\delta_u \mathcal{F} = [\mathcal{F}, u] \qquad \text{(Abelian: } \delta_u \mathcal{F} = 0\text{)}$$

The transformation of the connection one-form gives the definition of the **covariant derivative acting on zero-forms**:

$$\delta_u \mathscr{A}_\mu^a = (\mathcal{D}_\mu u)^a$$

$$\delta_u \mathcal{A} = du + [\mathcal{A}, u] \quad \longrightarrow \quad Du = du + [\mathcal{A}, u] \quad \longrightarrow \quad \delta_u \mathcal{A} = Du$$

On a **general Lie-algebra valued adjoint r-form**, the **covariant derivative** is defined by

$$D\omega_r \equiv d\omega_r + \mathcal{A}\omega_r - (-1)^r \omega_r \mathcal{A}$$

which satisfies the same **Leibniz rule** as the differential (Exercise)

$$D(\omega_s \eta_s) = (D\omega_r)\eta_s + (-1)^r \omega_r (D\eta_s)$$

and it is indeed **covariant** (Exercise)

$$D_g(g^{-1}\omega_r g) = g^{-1}(D\omega_r) g$$

constructed with $\mathcal{A}_g$

When **computing traces of forms**, one has to take into account their noncommutative character in applying the **cyclic property**

$$\mathrm{Tr}\,(\omega_r \eta_s) = (-1)^{rs}\mathrm{Tr}\,(\eta_s \omega_r)$$

or in general

$$\mathrm{Tr}\,(\omega_r \eta_{s_1} \ldots \eta_{s_n}) = (-1)^{r(s_1+\ldots+s_n)}\mathrm{Tr}\,(\eta_{s_1} \ldots \eta_{s_n} \omega_r)$$

For example, in the case of the **covariant derivative**

$$D\omega_r \equiv d\omega_r + \mathcal{A}\omega_r - (-1)^r \omega_r \mathcal{A}$$

$$\mathrm{Tr}\,D\omega_r = \mathrm{Tr}\,d\omega_r + \mathrm{Tr}\,(\mathcal{A}\omega_r) - (-1)^r\mathrm{Tr}\,(\omega_r\mathcal{A}) = \mathrm{Tr}\,d\omega_r = d\mathrm{Tr}\,\omega_r$$

Thus, the **trace** of a **covariant derivative** is an **exact form**.

Invariant polynomials

In an **even-dimensional space** $D=2m$, we can define the **invariant polynomial** associated with a gauge connection as

$$\mathcal{P}(\mathcal{F})=\sum_{nj\leq m}c_{n,j}\Big(\mathrm{Tr}\,\mathcal{F}^n\Big)^j$$

- **Invariant polynomials are gauge invariant:** we look at the single trace 2n-form

$$\mathrm{Tr}\,\mathcal{F}^n\longrightarrow\mathrm{Tr}\left(g^{-1}\mathcal{F}^n g\right)=\mathrm{Tr}\left(\mathcal{F}^n g g^{-1}\right)=\mathrm{Tr}\,\mathcal{F}^n$$

- **Invariant polynomials are closed forms:** computing the exterior differential

$$d\mathrm{Tr}\,\mathcal{F}^n=\mathrm{Tr}\left(d\mathcal{F}\mathcal{F}\ldots\mathcal{F}+\mathcal{F}d\mathcal{F}\ldots\mathcal{F}+\ldots+\mathcal{F}\mathcal{F}\ldots d\mathcal{F}\right)$$

$$=n\mathrm{Tr}\left(d\mathcal{F}\mathcal{F}^{n-1}\right)$$

$$d\mathrm{Tr}\,\mathcal{F}^n = n\mathrm{Tr}\left(d\mathcal{F}\mathcal{F}^{n-1}\right)$$

On each terms we can apply the **Bianchi identity** $d\mathcal{F} = \mathcal{F}\mathcal{A} - \mathcal{A}\mathcal{F}$

$$\begin{aligned} d\mathrm{Tr}\,\mathcal{F}^n &= n\mathrm{Tr}\left(\mathcal{F}\mathcal{A}\mathcal{F}^{n-1} - \mathcal{A}\mathcal{F}^n\right) \\ &= n\mathrm{Tr}\left(\mathcal{A}\mathcal{F}^n\right) - n\mathrm{Tr}\left(\mathcal{A}\mathcal{F}^n\right) = 0 \end{aligned}$$

With this, we have shown **two important properties** of the invariant polynomials

$$\delta_u \mathrm{Tr}\,\mathcal{F}^n = 0 \quad \Longrightarrow \quad \delta_u \mathcal{P}(\mathcal{F}) = 0$$

and

$$d\mathrm{Tr}\,\mathcal{F}^n = 0 \quad \Longrightarrow \quad d\mathcal{P}(\mathcal{F}) = 0$$

$$d\mathrm{Tr}\,\mathcal{F}^n = 0$$

Using this property and the Poincaré lemma, we conclude that the **locally**

$$\mathrm{Tr}\,\mathcal{F}^n = d\omega^0_{2n-1}(\mathcal{A})$$

where $\omega^0_{2n-1}(\mathcal{A})$ is the **Chern-Simons form**.

But since

$$\begin{aligned} 0 = \delta_u \mathrm{Tr}\,\mathcal{F}^n &= \mathrm{Tr}\,\mathcal{F}^n_{1+u} - \mathrm{Tr}\,\mathcal{F}^n = d\omega^0_{2n-1}(\mathcal{A}_{1+u}) - d\omega^0_{2n-1}(\mathcal{A}) \\ &= d\Big[\omega^0_{2n-1}(\mathcal{A}_{1+u}) - \omega^0_{2n-1}(\mathcal{A})\Big] = d\delta_u\omega^0_{2n-1}(\mathcal{A}) \end{aligned}$$

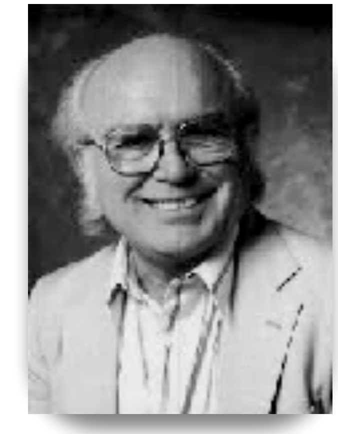
the gauge variation of the Chern-Simons form is also closed, and **locally** exact

$$d\delta_u\omega^0_{2n-1}(\mathcal{A}) = 0 \quad \xrightarrow{\text{locally}} \quad \delta_u\omega^0_{2n-1}(\mathcal{A}) = d\omega^1_{2n-2}(u,\mathcal{A})$$

Back to the anomaly

$$\mathrm{Tr}\,\mathcal{F}^n = d\omega^0_{2n-1}(\mathcal{A})$$

$$\delta_u\omega^0_{2n-1}(\mathcal{A}) = d\omega^1_{2n-2}(u,\mathcal{A})$$



Bruno Zumino
(1923-2014)

With these ingredients it is possible to construct a **nontrivial solution** to the **Wess-Zumino consistency condition**.

Let us take a (2n-1)-dimensional **ball** D_{2n-1} with $\partial D_{2n-1} = S^{2n-2}$ and write the integral

$$I[\mathcal{A}] = \int_{D_{2n-1}} \omega^0_{2n-1}(\mathcal{A})$$

Its gauge variation is given by

$$\delta_u I[\mathcal{A}] = \int_{D_{2n-1}} \delta_u\omega^0_{2n-1}(\mathcal{A}) = \int_{D_{2n-1}} d\omega^1_{2n-2}(u,\mathcal{A}) = \int_{S^{2n-2}} \omega^1_{2n-2}(u,\mathcal{A})$$

$$I[\mathcal{A}] = \int_{D_{2n-1}} \omega^0_{2n-1}(\mathcal{A}) \qquad\qquad \delta_u I[\mathcal{A}] = \int_{S^{2n-2}} \omega^1_{2n-2}(u,\mathcal{A})$$

We **identify** the anomaly in **2n-2 dimensions** as

$$\int_{S^{2n-2}} u^a \mathscr{G}_a[\mathcal{A}] = -c_n \int_{S^{2n-2}} \omega^1_{2n-2}(u,\mathcal{A})$$

being the **variation** of a functional it automatically **solves** the **Wess-Zumino consistency equation**.

Moreover, the (2n-1)-dimensional integral $I[\mathcal{A}]$ is **nonlocal** in the **physical (2n-2)-dimensional space**.

$\omega^1_{2n-2}(u,\mathcal{A})$ is a **nontrivial solution** to the **Wess-Zumino equations**

$$I[\mathcal{A}] = \int_{D_{2n-1}} \omega^0_{2n-1}(\mathcal{A}) \qquad\qquad \delta_u I[\mathcal{A}] = \int_{S^{2n-2}} \omega^1_{2n-2}(u, \mathcal{A})$$

We **identify** the anomaly in **2n-2 dimensions** as

normalization constant

$$\int_{S^{2n-2}} u^a \mathscr{G}_a[\mathcal{A}] = -c_n \int_{S^{2n-2}} \omega^1_{2n-2}(u, \mathcal{A})$$

being the **variation** of a functional it automatically **solves** the **Wess-Zumino consistency equation**.

Moreover, the (2n-1)-dimensional integral $I[\mathcal{A}]$ is **nonlocal** in the **physical (2n-2)-dimensional space**.

$\Downarrow$

$\omega^1_{2n-2}(u, \mathcal{A})$ is a **nontrivial solution** to the **Wess-Zumino equations**

For a **single left-handed fermion in D=2n-2 dimensions**, the normalization constant can be computed (e.g. using diagrammatics)

$$c_n = \frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}$$

The **fermion effective action** is given by

$$\boxed{\Gamma[\mathcal{A}] = \frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{D_{2n-1}} \omega^0_{2n-1}(\mathcal{A})}$$

while the anomaly is

$$\boxed{\int_{S^{2n-2}} u^a \mathscr{G}_a[\mathcal{A}] = -\frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{S^{2n-2}} \omega^1_{2n-2}(u,\mathcal{A})}$$

Local ambiguities correspond to adding a **total differential** to the Chern-Simons form

local in D=2n-2

$$\Gamma[\mathcal{A}] = \frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{D_{2n-1}} \left[\omega^0_{2n-1}(\mathcal{A}) + d\alpha_{2n-2}(\mathcal{A})\right]$$

Let us particularize the analysis to **D=4** (**n=3**). The **relevant anomaly polynomial** es

$$\mathcal{P}(\mathcal{F}) = \mathrm{Tr}\,\mathcal{F}^3$$

To compute the Chern-Simons form we use an **homotopy formula**. Consider the **family of connections**

$$\mathcal{A}_t = t\mathcal{A} \qquad \text{with} \qquad 0 \leq t \leq 1$$

with field strength

$$\mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t^2 = td\mathcal{A} + t^2\mathcal{A}^2 \quad \Longrightarrow \quad \mathcal{P}(\mathcal{F}_t) = \mathrm{Tr}\,\mathcal{F}_t^3$$

Differentiating with respect to the parameter

$$\frac{d}{dt}\mathrm{Tr}\,\mathcal{F}_t^3 = 3\mathrm{Tr}\left(\dot{\mathcal{F}}_t\mathcal{F}_t^2\right) = 3\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) + 3\mathrm{Tr}\left(\mathcal{A}_t\dot{\mathcal{A}}_t\mathcal{F}_t^2\right)$$

$$= 3\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) - 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right)$$

$$\frac{d}{dt}\mathrm{Tr}\,\mathcal{F}_t^3 = 3\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) - 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right)$$

Applying the **Bianchi identity** $d\mathcal{F}_t = d\mathcal{A}_t\mathcal{A}_t - \mathcal{A}_t d\mathcal{A}_t$

$$\begin{aligned}
\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) &= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t d\mathcal{F}_t\mathcal{F}_t\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t d\mathcal{F}_t\right) \\
&= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t d\mathcal{A}_t\mathcal{A}_t\mathcal{F}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t d\mathcal{A}_t\mathcal{F}_t\right) \\
&+ \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t d\mathcal{A}_t\mathcal{A}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t d\mathcal{A}_t\right)
\end{aligned}$$

and using $d\mathcal{A}_t = \mathcal{F}_t - \mathcal{A}_t^2$

$$\begin{aligned}
\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) &= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t\mathcal{F}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t^3\mathcal{F}_t\right) \\
&- \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t^3\mathcal{F}_t\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right) \\
&- \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t^3\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t\mathcal{F}_t\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t^3\right)
\end{aligned}$$

$$\frac{d}{dt}\mathrm{Tr}\,\mathcal{F}_t^3 = 3\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) - 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right)$$

Applying the **Bianchi identity** $d\mathcal{F}_t = d\mathcal{A}_t\mathcal{A}_t - \mathcal{A}_t d\mathcal{A}_t$

$$\begin{aligned}
\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) &= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t d\mathcal{F}_t\mathcal{F}_t\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t d\mathcal{F}_t\right) \\
&= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t d\mathcal{A}_t\mathcal{A}_t\mathcal{F}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t d\mathcal{A}_t\mathcal{F}_t\right) \\
&\quad + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t d\mathcal{A}_t\mathcal{A}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t d\mathcal{A}_t\right)
\end{aligned}$$

and using $d\mathcal{A}_t = \mathcal{F}_t - \mathcal{A}_t^2$

$$\begin{aligned}
\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) &= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \cancel{\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t\mathcal{F}_t\right)} - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t^3\mathcal{F}_t\right) \\
&\quad - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t^3\mathcal{F}_t\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right) \\
&\quad - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t^3\right) - \cancel{\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t\mathcal{F}_t\right)} + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t^3\right)
\end{aligned}$$

$$\frac{d}{dt}\mathrm{Tr}\,\mathcal{F}_t^3 = 3\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) - 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right)$$

Applying the **Bianchi identity** $d\mathcal{F}_t = d\mathcal{A}_t\mathcal{A}_t - \mathcal{A}_t d\mathcal{A}_t$

$$\begin{aligned}
\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) &= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t d\mathcal{F}_t\mathcal{F}_t\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t d\mathcal{F}_t\right) \\
&= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t d\mathcal{A}_t\mathcal{A}_t\mathcal{F}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t d\mathcal{A}_t\mathcal{F}_t\right) \\
&\quad + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t d\mathcal{A}_t\mathcal{A}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t d\mathcal{A}_t\right)
\end{aligned}$$

and using $d\mathcal{A}_t = \mathcal{F}_t - \mathcal{A}_t^2$

$$\begin{aligned}
\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) &= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\cancel{\mathcal{F}_t\mathcal{A}_t}\mathcal{F}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\cancel{\mathcal{A}_t^3}\mathcal{F}_t\right) \\
&\quad - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\cancel{\mathcal{A}_t^3\mathcal{F}_t}\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right) \\
&\quad - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t^3\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\cancel{\mathcal{F}_t\mathcal{A}_t}\mathcal{F}_t\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t^3\right)
\end{aligned}$$

$$\frac{d}{dt}\mathrm{Tr}\,\mathcal{F}_t^3 = 3\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) - 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right)$$

Applying the **Bianchi identity** $d\mathcal{F}_t = d\mathcal{A}_t\mathcal{A}_t - \mathcal{A}_t d\mathcal{A}_t$

$$\begin{aligned}
\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) &= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t d\mathcal{F}_t\mathcal{F}_t\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t d\mathcal{F}_t\right) \\
&= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t d\mathcal{A}_t\mathcal{A}_t\mathcal{F}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t d\mathcal{A}_t\mathcal{F}_t\right) \\
&+ \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t d\mathcal{A}_t\mathcal{A}_t\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t d\mathcal{A}_t\right)
\end{aligned}$$

and using $d\mathcal{A}_t = \mathcal{F}_t - \mathcal{A}_t^2$

$$\begin{aligned}
\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) &= d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + \cancel{\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t\mathcal{F}_t\right)} - \cancel{\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t^3\mathcal{F}_t\right)} \\
&- \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) + \cancel{\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t^3\mathcal{F}_t\right)} + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right) \\
&- \cancel{\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t^3\right)} - \cancel{\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t\mathcal{F}_t\right)} + \cancel{\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t\mathcal{A}_t^3\right)}
\end{aligned}$$

$$\frac{d}{dt}\mathrm{Tr}\,\mathcal{F}_t^3 = 3\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) + 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) - 3\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right)$$

$$\mathrm{Tr}\left(d\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) = d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right) - \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{A}_t\mathcal{F}_t^2\right) + \mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\mathcal{A}_t\right)$$

$$\Downarrow$$

$$\frac{d}{dt}\mathrm{Tr}\,\mathcal{F}_t^3 = 3d\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right)$$

Now we can **integrate** over the parameter t (remember $\mathcal{A}_t = t\mathcal{A}$)

$$\mathrm{Tr}\,\mathcal{F}^3 = 3d\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right)$$

so we can identify the **Chern-Simons form** as

$$\omega_5^0(\mathcal{A}) = 3\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right)$$

$$\omega_5^0(\mathcal{A}) = 3\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right)$$

We can carry out the integral **explicitly** by using

$$\mathcal{A}_t = t\mathcal{A} \qquad\qquad \mathcal{F}_t = td\mathcal{A} + t^2\mathcal{A}^2$$

$$\omega_5^0(\mathcal{A}) = 3\int_0^1 dt\,\mathrm{Tr}\left[t^2\mathcal{A}(d\mathcal{A})^2 + t^3\mathcal{A}d\mathcal{A}\mathcal{A}^2 + t^3\mathcal{A}^3d\mathcal{A} + t^4\mathcal{A}^5\right]$$

$$\omega_5^0(\mathcal{A}) = \mathrm{Tr}\left[\mathcal{A}(d\mathcal{A})^2 + \frac{3}{2}\mathcal{A}^3d\mathcal{A} + \frac{3}{5}\mathcal{A}^5\right]$$

or in terms of the **field-strength**

$$\boxed{\omega_5^0(\mathcal{A},\mathcal{F}) = \mathrm{Tr}\left(\mathcal{A}\mathcal{F}^2 - \frac{1}{2}\mathcal{A}^3\mathcal{F} + \frac{1}{10}\mathcal{A}^5\right)}$$

$$\omega_5^0(\mathcal{A},\mathcal{F}) = \mathrm{Tr}\,\left(\mathcal{A}\mathcal{F}^2 - \frac{1}{2}\mathcal{A}^3\mathcal{F} + \frac{1}{10}\mathcal{A}^5\right)$$

Taking a gauge variation of this expression,

$$\delta_u\omega_5^0(\mathcal{A},\mathcal{F}) = \mathrm{Tr}\,\left(\delta_u\mathcal{A}\mathcal{F}^2 + \delta_u\mathcal{F}\mathcal{F}\mathcal{A} + \delta_u\mathcal{F}\mathcal{A}\mathcal{F} - \frac{1}{2}\delta_u\mathcal{A}\mathcal{A}^2\mathcal{F}\right.$$

$$\left. - \frac{1}{2}\delta_u\mathcal{A}\mathcal{A}\mathcal{F}\mathcal{A} - \frac{1}{2}\delta_u\mathcal{A}\mathcal{F}\mathcal{A}^2 - \frac{1}{2}\delta_u\mathcal{F}\mathcal{A}^3 + \frac{1}{10}\delta_u\mathcal{A}\mathcal{A}^4\right)$$

$$\delta_u\mathcal{A} = du + [\mathcal{A}, u] \qquad \Downarrow \qquad \delta_u\mathcal{F} = [\mathcal{F}, u]$$

$$\delta_u\omega_5^0(\mathcal{A},\mathcal{F}) = \mathrm{Tr}\,\left[(Du)\left(\mathcal{F}^2 - \frac{1}{2}\mathcal{A}^2\mathcal{F} - \frac{1}{2}\mathcal{A}\mathcal{F}\mathcal{A} - \frac{1}{2}\mathcal{F}\mathcal{A}^2 + \frac{1}{2}\mathcal{A}^4\right)\right]$$

and writing it in terms of the gauge potential

$$\delta_u\omega_5^0(\mathcal{A},\mathcal{F}) = \mathrm{Tr}\,\left[(Du)d\left(\mathcal{A}d\mathcal{A} + \frac{1}{2}\mathcal{A}^3\right)\right]$$

$$\delta_u \omega_5^0(\mathcal{A}) = \mathrm{Tr}\left[(Du)d\left(\mathcal{A}d\mathcal{A}+\frac{1}{2}\mathcal{A}^3\right)\right]$$

$$D\left[d\left(\mathcal{A}d\mathcal{A}+\frac{1}{2}\mathcal{A}^3\right)\right]=0$$

$$\delta_u \omega_5^0(\mathcal{A}) = d\mathrm{Tr}\left[ud\left(\mathcal{A}d\mathcal{A}+\frac{1}{2}\mathcal{A}^3\right)\right]$$

$$\delta_u \omega_5^0 = d\omega_4^1$$

$$\boxed{\omega_4^1(u,\mathcal{A}) = \mathrm{Tr}\left[ud\left(\mathcal{A}d\mathcal{A}+\frac{1}{2}\mathcal{A}^3\right)\right]}$$

$$\omega_5^0(\mathcal{A},\mathcal{F}) = \mathrm{Tr}\,\left(\mathcal{A}\mathcal{F}^2 - \frac{1}{2}\mathcal{A}^3\mathcal{F} + \frac{1}{10}\mathcal{A}^5\right)$$

$$\omega_4^1(u,\mathcal{A}) = \mathrm{Tr}\,\left[ud\left(\mathcal{A}d\mathcal{A} + \frac{1}{2}\mathcal{A}^3\right)\right]$$

We can now compute the **anomalous effective action**

$$\Gamma[\mathcal{A}] = -\frac{i}{24\pi^2}\int_{D_5}\mathrm{Tr}\,\left(\mathcal{A}\mathcal{F}^2 - \frac{1}{2}\mathcal{A}^3\mathcal{F} + \frac{1}{10}\mathcal{A}^5\right)$$

nonlocal!

which gives the anomaly

$$\int_{S^4} u^a \mathscr{G}_a[\mathcal{A}] = \frac{i}{24\pi^2}\int_{S^4}\mathrm{Tr}\,\left[ud\left(\mathcal{A}d\mathcal{A} + \frac{1}{2}\mathcal{A}^3\right)\right]$$

$$= \frac{i}{24\pi^2}\int_{S^4} u^a\mathrm{Tr}\,\left[T^a d\left(\mathcal{A}d\mathcal{A} + \frac{1}{2}\mathcal{A}^3\right)\right]$$

$$\omega_5^0(\mathcal{A},\mathcal{F}) = \mathrm{Tr}\left(\mathcal{A}\mathcal{F}^2 - \frac{1}{2}\mathcal{A}^3\mathcal{F} + \frac{1}{10}\mathcal{A}^5\right)$$

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$$= \frac{i}{24\pi^2}\int_{S^4} u^a\mathrm{Tr}\left[T^a d\left(\mathcal{A}d\mathcal{A} + \frac{1}{2}\mathcal{A}^3\right)\right]$$

Euclidean space

This gives the **consistent anomaly** in four-dimensions ($\mathcal{A} = -i\mathscr{A}_\mu dx^\mu$)

$$\mathscr{G}_a[\mathscr{A}] = \frac{i}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[T^a\partial_\mu\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta - \frac{i}{2}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right)\right]$$

which **reproduces**, in **Euclidean space**, the **Bardeen anomaly** for a left-handed fermion (in the notation used back there)

$$\langle(\mathcal{D}_\mu J^\mu_L)^a\rangle_{\mathscr{L}} = \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[T^a\partial_\mu\left(\mathscr{L}_\nu\partial_\alpha\mathscr{L}_\beta - \frac{i}{2}\mathscr{L}_\nu\mathscr{L}_\alpha\mathscr{L}_\beta\right)\right]$$

For a **right-handed fermion**, we have the a similar contribution but with **opposite global sign**:

$$\Gamma[\mathcal{A}] = \mp\frac{i}{24\pi^2}\int_{D_5}\mathrm{Tr}\left(\mathcal{A}\mathcal{F}^2 - \frac{1}{2}\mathcal{A}^3\mathcal{F} + \frac{1}{10}\mathcal{A}^5\right)$$

− left-handed
+ right-handed

$$\int_{S^4} u^a\mathscr{G}_a[\mathcal{A}] = \pm\frac{i}{24\pi^2}\int_{S^4} u^a\mathrm{Tr}\left[T^a d\left(\mathcal{A}d\mathcal{A} + \frac{1}{2}\mathcal{A}^3\right)\right]$$

+ left-handed
− right-handed

To **summarize**: to find the **chiral anomaly** in dimension $D = 2n - 2$

- Construct the **anomaly polynomial** in dimension $D + 2 = 2n$

$$\mathcal{P}(\mathcal{F}) = \frac{1}{n!} \frac{i^n}{(2n)^{n-1}} \mathrm{Tr}\, \mathcal{F}^n$$

- The (nonlocal) **anomalous effective action** is given by the integral of the corresponding **Chern-Simons form** in dimension $D + 1 = 2n - 1$

$$\mathrm{Tr}\, \mathcal{F}^n = d\omega^0_{2n-1}(\mathcal{A}) \quad \Longrightarrow \quad \Gamma[\mathcal{A}] = \frac{1}{n!} \frac{i^n}{(2\pi)^{n-1}} \int_{D_{2n-1}} \omega^0_{2n-1}(\mathcal{A})$$

up to the addition of an exact (2n-1)-form (**local counterterm**)

- The (local) **anomaly** is given in terms of $\delta_u \omega^0_{2n-1}(\mathcal{A}) = d\omega^1_{2n-2}(u, \mathcal{A})$ by

$$\int_{S^{2n-2}} u^a \mathscr{G}_a[\mathcal{A}] = -\frac{1}{n!} \frac{i^n}{(2\pi)^{n-1}} \int_{S^{2n-2}} \omega^1_{2n-2}(u, \mathcal{A})$$

The BRST formulation and the descent equations

BRST transformations

Acting on the gauge theory fields, we define the action of the **BRST operator** s with a **odd adjoint (zero-form) parameter** v

anticommute with odd-rank forms

$$s\mathcal{A} = -Dv \qquad \text{(with } Dv = dv + \{\mathcal{A}, v\})$$

$$s\mathcal{F} = -[v, \mathcal{F}]$$

$$sv = -v^2$$

We assign **ghost numbers**:

$$\mathrm{gh}(\mathcal{A}) = 0$$

$$\mathrm{gh}(\mathcal{F}) = 0$$

$$\mathrm{gh}(v) = 1$$

with s **increasing** the ghost number in **one unit**.

$$\mathrm{gh}(s\mathcal{O}) = \mathrm{gh}(\mathcal{O}) + 1$$

BRST transformations

Consistency of these transformations requires

$$sd + ds = 0$$

Indeed,

$$s\mathcal{F} = s\Big(d\mathcal{A} + \mathcal{A}^2\Big) = sd\mathcal{A} + (s\mathcal{A})\mathcal{A} - \mathcal{A}(s\mathcal{A})$$

$$= -d(s\mathcal{A}) - (Dv)\mathcal{A} + (Dv)\mathcal{A} = d(Dv) - (Dv)\mathcal{A} + \mathcal{A}(Dv)$$

$$= d\mathcal{A}v - \mathcal{A}dv + dv\mathcal{A} - vd\mathcal{A} - dv\mathcal{A} + \mathcal{A}dv - v\mathcal{A}^2 - \mathcal{A}v\mathcal{A} + \mathcal{A}^2v + \mathcal{A}v\mathcal{A}$$

$sd = -ds$

BRST transformations

Consistency of these transformations requires

$$sd + ds = 0$$

Indeed,

$$s\mathcal{F} = s\Big(d\mathcal{A} + \mathcal{A}^2\Big) = sd\mathcal{A} + (s\mathcal{A})\mathcal{A} - \mathcal{A}(s\mathcal{A})$$

$$= -d(s\mathcal{A}) - (Dv)\mathcal{A} + (Dv)\mathcal{A} = d(Dv) - (Dv)\mathcal{A} + \mathcal{A}(Dv)$$

$$= d\mathcal{A}v - \cancel{\mathcal{A}dv} + \cancel{dv\mathcal{A}} - vd\mathcal{A} - \cancel{dv\mathcal{A}} + \cancel{\mathcal{A}dv} - v\mathcal{A}^2 - \cancel{\mathcal{A}v\mathcal{A}} + \mathcal{A}^2v + \cancel{\mathcal{A}v\mathcal{A}}$$

$sd = -ds$

$$s\mathcal{F} = d\mathcal{A}v - vd\mathcal{A} - v\mathcal{A}^2 + \mathcal{A}^2v$$

$$= -[v, d\mathcal{A} + \mathcal{A}^2]$$

$$= -[v, \mathcal{F}]$$

BRST transformations

The BRST operations is **nihilpotent**

$$s^2 = 0$$

Let us **check it** explicitly:

$$\begin{aligned} s^2\mathcal{A} &= -s(dv + \mathcal{A}v + v\mathcal{A}) = d(sv) - (s\mathcal{A})v + \mathcal{A}(sv) + (sv)\mathcal{A} - v(s\mathcal{A}) \\ &= -d(v^2) + (dv + \mathcal{A}v + v\mathcal{A})v - \mathcal{A}v^2 + v^2\mathcal{A} - v(dv + \mathcal{A}v + v\mathcal{A}) = 0 \end{aligned}$$

$$\begin{aligned} s^2\mathcal{F} &= -s(v\mathcal{F} - \mathcal{F}v) = -(sv)\mathcal{F} + v(s\mathcal{F}) + (s\mathcal{F})v + \mathcal{F}(sv) \\ &= v^2\mathcal{F} - v(v\mathcal{F} - \mathcal{F}v) - (v\mathcal{F} - \mathcal{F}v)v - \mathcal{F}v^2 = 0 \end{aligned}$$

$$s^2 v = -s(v^2) = -(sv)v + v(sv) = v^3 - v^3 = 0$$

BRST transformations

We get now Stora's **Russian formula**

$$\boxed{(d+s)(\mathcal{A}+v)+(\mathcal{A}+v)^2 = d\mathcal{A}+\mathcal{A}^2}$$

$$\begin{aligned}
(d+s)(\mathcal{A}+v)+(\mathcal{A}+v)^2 &= d\mathcal{A}+\mathcal{A}^2+dv+s\mathcal{A}+sv+\mathcal{A}v+v\mathcal{A}+v^2 \\
&= d\mathcal{A}+\mathcal{A}^2+\Big(s\mathcal{A}+dv+\mathcal{A}v+v\mathcal{A}\Big)+\Big(sv+v^2\Big) \\
&= d\mathcal{A}+\mathcal{A}^2+\Big(s\mathcal{A}+Dv\Big)+\Big(sv+v^2\Big) \\
&= d\mathcal{A}+\mathcal{A}^2
\end{aligned}$$

$$s\mathcal{A}=-Dv \qquad sv=-v^2$$

This means that $\mathcal{F}$ is **left invariant** by the **replacement**

$$d \longrightarrow d+s$$

$$\mathcal{A} \longrightarrow \mathcal{A}+v$$

Let us apply the **BRST** formalism to the problem of **anomalies**. If we write the transformations in components

$$s\mathscr{A}_\mu^a = -\mathcal{D}_\mu v^a$$

we find that acting of the effective action

$$s\Gamma[\mathscr{A}] = \int d^D x\,[s\mathscr{A}_\mu^a(x)]\frac{\delta}{\delta\mathscr{A}_\mu^a(x)}\Gamma[\mathscr{A}] = -\int d^D x\,[\mathcal{D}_\mu v(x)]^a\langle J^{\mu a}(x)\rangle_{\mathscr{A}}$$

$$= \int d^D x\,v^a(x)\Big[\mathcal{D}_\mu\langle J^{\mu a}(x)\rangle_{\mathscr{A}}\Big]^a = \int d^D x\,v^a(x)\mathscr{G}_a[\mathscr{A}(x)]$$

so the **BRST** transformations of the action gives the **anomaly**

$$\boxed{s\Gamma[\mathcal{A}] = \int v^a\mathscr{G}_a[\mathcal{A}]}$$

$$s\Gamma[\mathcal{A}] = \int v^a \mathscr{G}_a[\mathcal{A}] \equiv \int \mathscr{G}^1[v, \mathcal{A}]$$

The **Wess-Zumino consistency condition** now takes a **extremely simple form**

$$s^2\Gamma[\mathcal{A}] = 0$$

$$\int s\mathscr{G}^1[v, \mathcal{A}] = 0$$

It is obvious that **any** anomaly obtained as the **BRST variation** of a **functional** automatically satisfy the **consistency condition**.

Only **nontrivial** (i.e. **nonlocal**) solutions to the Wess-Zumino equations **can give anomalies**.

Let us apply now the Russian formula to the **anomaly polynomial**:

$$\mathrm{Tr}\left[(d+s)(\mathcal{A}+v)+(\mathcal{A}+v)^2\right]^n = \mathrm{Tr}\,\mathcal{F}^n$$

and since $\mathrm{Tr}\,\mathcal{F}^n = d\omega^0_{2n-1}(\mathcal{A},\mathcal{F})$

all $d\mathcal{A}$'s have been written in terms of $\mathcal{F}$'s

$$(d+s)\omega^0_{2n-1}(\mathcal{A}+v,\mathcal{F}) = d\omega^0_{2n-1}(\mathcal{A},\mathcal{F})$$

now we can expand the Chern-Simons form in **powers of *v***

ghost number

$$\omega^0_{2n-1}(\mathcal{A}+v,\mathcal{F}) = \omega^0_{2n-1}(\mathcal{A},\mathcal{F}) + \omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) + \ldots + \omega^{2n-1}_0(v,\mathcal{A},\mathcal{F})$$

and equal **order by order** in the **ghost number expansion**. At zeroth order, we have trivially

$$d\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) = d\omega^0_{2n-1}(\mathcal{A},\mathcal{F})$$

$$(d+s)\omega^0_{2n-1}(\mathcal{A}+v,\mathcal{F}) = d\omega^0_{2n-1}(\mathcal{A},\mathcal{F})$$

$$\omega^0_{2n-1}(\mathcal{A}+v,\mathcal{F}) = \omega^0_{2n-1}(\mathcal{A},\mathcal{F}) + \omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) + \ldots + \omega^{2n-1}_0(v,\mathcal{A},\mathcal{F})$$

At **first order**, we have a nontrivial identity

$$s\omega^0_{2n-2}(\mathcal{A},\mathcal{F}) + d\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) = 0$$

while at the **following orders** we find

$$s\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) + d\omega^2_{2n-3}(v,\mathcal{A},\mathcal{F}) = 0$$

$$s\omega^2_{2n-3}(v,\mathcal{A},\mathcal{F}) + d\omega^3_{2n-4}(v,\mathcal{A},\mathcal{F}) = 0$$

$$\vdots$$

$$s\omega^m_{2n-m-1}(v,\mathcal{A},\mathcal{F}) + d\omega^{m+1}_{2n-m-2}(v,\mathcal{A},\mathcal{F}) = 0$$

up to $m = 2n-1$

We have arrived at the **Stora-Zumino descent equations**

$$\mathrm{Tr}\,\mathcal{F}^n - d\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) = 0$$

$$s\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) + d\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) = 0$$

$$s\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) + d\omega^2_{2n-3}(v,\mathcal{A},\mathcal{F}) = 0$$

$$\vdots$$

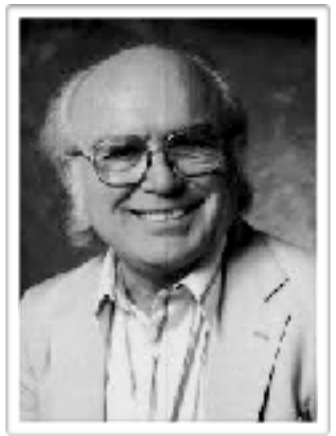
$$s\omega^m_{2n-m-1}(v,\mathcal{A},\mathcal{F}) + d\omega^{m+1}_{2n-m-2}(v,\mathcal{A},\mathcal{F}) = 0$$

$$\vdots$$

$$s\omega^{2n-2}_1(v,\mathcal{A},\mathcal{F}) + d\omega^{2n-1}_0(v,\mathcal{A},\mathcal{F}) = 0$$

$$s\omega^{2n-1}_0(v,\mathcal{A},\mathcal{F}) = 0$$

Raymond Stora
(1930-2015)



Bruno Zumino
(1923-2014)

The Stora-Zumino descent equations give nontrivial solutions to the Wess-Zumino equations. We start with the **nonlocal effective action**

$$\Gamma[\mathcal{A}] = \frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{D_{2n-1}} \omega^0_{2n-1}(\mathcal{A},\mathcal{F})$$

Using the **second descent equation**, we have

$$s\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) + d\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) = 0$$

$$\begin{aligned} s\Gamma[\mathcal{A}] &= \frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{D_{2n-1}} s\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) \\ &= -\frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{D_{2n-1}} d\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) \\ &= -\frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{S^{2n-2}} \omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) \end{aligned}$$

$$s\Gamma[\mathcal{A}] = -\frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{S^{2n-2}}\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F})$$

The **anomaly** is then given by

$$\int_{S^{2n-2}}\mathscr{G}^1[v,\mathcal{A}] = -\frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{S^{2n-2}}\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F})$$

Using now the **third** descent equation

$$s\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) + d\omega^2_{2n-3}(v,\mathcal{A},\mathcal{F}) = 0$$

we see that it satisfies the Wess-Zumino **consistency condition**

$$\partial S^{2n-2} = \emptyset$$

$$\begin{aligned}\int_{S^{2n-2}}s\mathscr{G}^1[v,\mathcal{A}] &= -\frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{S^{2n-2}}s\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) \\ &= -\frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{S^{2n-2}}d\omega^2_{2n-3}(v,\mathcal{A},\mathcal{F}) = 0\end{aligned}$$

Being derived from a **nonlocal functional**, it is a **nontrivial solution!**

Ambiguities in the **anomaly** are related to the structure of the descent equations. A **generic solution** to the **consistency conditions** has the **structure**

$$\int_{S^{2n-2}} s\Big(\omega^1_{2n-2} + s\alpha^0_{2n-2} + d\beta^1_{2n-3}\Big) = 0$$

From the **third** descent equation

$$s\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) + d\omega^2_{2n-3}(v,\mathcal{A},\mathcal{F}) = 0$$

if follows that ω^1_{2n-2} is defined up to a **BRST-exact term**

$$\omega^1_{2n-2} \longrightarrow \omega^1_{2n-2} + s\alpha^1_{2n-3}$$

Using the **second** descent equation

$$s\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) + d\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) = 0$$

we see that β^1_{2n-3} corresponds to the ambiguity

$$\omega^1_{2n-2} \longrightarrow \omega^1_{2n-2} + d\beta^1_{2n-3}$$

Using the **descent equations** we can see how **BRST-exact shifts** in the **anomaly** are associated with the addition of **local counterterms** to the **effective action** functional.

The **first equation**

$$\mathrm{Tr}\,\mathcal{F}^n - d\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) = 0$$

remains **unchanged** under the addition of a **local counterterm**

$$\omega^0_{2n-1} \longrightarrow \omega^0_{2n-1} + d\gamma^0_{2n-2}$$

Looking however at the **second equation**

$$s\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) + d\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) = 0$$

we see that this change can be cancelled by a **BRST-exact shift** in the anomaly

$$\omega^1_{2n-2} \longrightarrow \omega^1_{2n-2} + s\gamma^0_{2n-2} \qquad\qquad (ds + sd = 0)$$

Computing the anomaly

We would like to find a simple way to compute **anomalies** (Chern-Simons forms and their descendants) in **any dimension**.

Let us introduce a **family of connections** $\mathcal{A}_t$ depending on a number of **parameters** taking values on a **domain T**

$$\mathcal{A}_{\boldsymbol{t}} \equiv \mathcal{A}_{t_1,t_2,\ldots} \qquad\qquad (t_1,t_2,\ldots)\in T$$

Define an **even substitution operator** ℓ_t replacing exterior differentials by differentials on the domain T

$$\ell_t \equiv d_t \frac{\partial}{\partial(d)} \qquad \text{with} \qquad d_t = \sum_{r=0}^{p+1} dt_r \frac{\partial}{\partial t_r}$$

Example: $\mathcal{A}_t = t\mathcal{A}_1 + (1-t)\mathcal{A}_2$ with $0\leq t\leq 1$

$$\ell_t\mathcal{A}_t = 0$$

$$\ell_t\mathcal{F}_t = d_t\mathcal{A}_t = dt\big(\mathcal{A}_1 - \mathcal{A}_2\big)$$

Let us consider now a **polynomial** of degree q in d_t

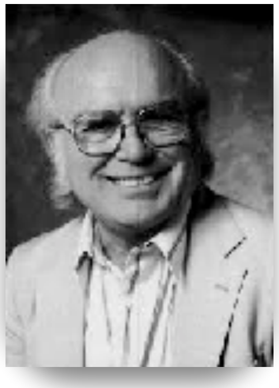
$$\mathscr{Q} \equiv \mathscr{Q}(\mathcal{A}_{\boldsymbol{t}}, \mathcal{F}_{\boldsymbol{t}}, d_t\mathcal{A}_{\boldsymbol{t}}, d_t\mathcal{F}_{\boldsymbol{t}})$$

It satisfies the **generalized transgression formula**

$$\boxed{\int_{\partial T} \frac{\ell_t^p}{p!}\mathscr{Q} = \int_T \frac{\ell_t^{p+1}}{(p+1)!} d\mathscr{Q} + (-1)^{p+q} d\int_T \frac{\ell_t^{p+1}}{(p+1)!}\mathscr{Q}}$$

Juan L. Mañes
(b. 1955)

Raymond Stora
(1930-2015)



Bruno Zumino
(1923-2014)

For the time being, let us apply it to the previous **example**

$$\mathcal{A}_t = t\mathcal{A}_1 + (1-t)\mathcal{A}_2 \qquad \text{with} \qquad p = 0$$

and take $\mathscr{Q}$ the **anomaly polynomial** $(q=0)$

$$\mathscr{Q} = \mathrm{Tr}\,\mathcal{F}_t^n$$

$$\Downarrow \qquad d\mathrm{Tr}\,\mathcal{F}_t^n = 0$$

$$\mathrm{Tr}\,\mathcal{F}_1^n - \mathrm{Tr}\,\mathcal{F}_2^n = d\int_T \ell_t \mathrm{Tr}\,\mathcal{F}_t^n$$

$$\mathrm{Tr}\,\mathcal{F}_1^n - \mathrm{Tr}\,\mathcal{F}_2^n = d\int_T \ell_t \mathrm{Tr}\,\mathcal{F}_t^n$$

Remembering that ℓ_t is an **even operator** and that

$$\ell_t \mathcal{F}_t = d_t \mathcal{A}_t = dt\big(\mathcal{A}_1 - \mathcal{A}_2\big)$$

we can compute the integrand to be

$$\ell_t \mathrm{Tr}\,\mathcal{F}_t^n = \mathrm{Tr}\left(\ell_t \mathcal{F}_t \mathcal{F}_t^{n-1} + \mathcal{F}_t \ell_t \mathcal{F}_t \mathcal{F}_t^{n-2} + \ldots + \mathcal{F}_t^{n-1} \ell_t \mathcal{F}_t\right)$$

$$= \mathrm{Tr}\left(d_t \mathcal{A}_t \mathcal{F}_t^{n-1} + \mathcal{F}_t d_t \mathcal{A}_t \mathcal{F}_t^{n-2} + \ldots + \mathcal{F}_t^{n-1} d_t \mathcal{A}_t\right)$$

$$= n\mathrm{Tr}\left(d_t \mathcal{A}_t \mathcal{F}_t^{n-1}\right)$$

and taking $\mathcal{A}_2 = 0$ and $\mathcal{A}_1 = \mathcal{A}$

$$\mathrm{Tr}\,\mathcal{F}^n = nd\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathcal{A}}_t \mathcal{F}_t^{n-1}\right) \qquad \text{with} \qquad \mathcal{A}_t = t\mathcal{A}$$

$$\mathrm{Tr}\,\mathcal{F}^n = nd\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^{n-1}\right)$$

From here, we readily read the general homotopy formula for the **Chern-Simons form** in any dimension

$$\boxed{\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) = n\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^{n-1}\right)}$$

where

$$\mathcal{A}_t = t\mathcal{A}$$

$$\mathcal{F}_t = td\mathcal{A} + t^2\mathcal{A}^2 = t\mathcal{F} + t(t-1)\mathcal{A}^2$$

This **generalizes** the expression obtained in **four dimensions** (n=3)

$$\omega^0_5(\mathcal{A},\mathcal{F}) = 3\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^2\right)$$

Example: the six-dimensional gauge anomaly (n = 4)

$$\omega_7^0(\mathcal{A},\mathcal{F}) = 4\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathcal{A}}_t\mathcal{F}_t^3\right) = 4\int_0^1 dt\,t^3\mathrm{Tr}\left\{\mathcal{A}\Big[\mathcal{F}+(t-1)\mathcal{A}^2\Big]^3\right\}$$

$$= 4\int_0^1 dt\,t^3\mathrm{Tr}\Big\{\mathcal{A}\Big[\mathcal{F}^3+(t-1)\Big(\mathcal{F}^2\mathcal{A}^2+\mathcal{F}\mathcal{A}^2\mathcal{F}+\mathcal{A}^2\mathcal{F}^2\Big)$$

$$+(t-1)^2\Big(\mathcal{F}\mathcal{A}^4+\mathcal{A}^2\mathcal{F}\mathcal{A}^2+\mathcal{A}^4\mathcal{F}\Big)+(t-1)^3\mathcal{A}^6\Big]\Big\}$$

and integrating over the parameter

$$\omega_7^0(\mathcal{A},\mathcal{F}) = \mathrm{Tr}\Big\{\Big[\mathcal{A}\mathcal{F}^3-\frac{1}{5}\Big(\mathcal{A}\mathcal{F}^2\mathcal{A}^2+\mathcal{A}\mathcal{F}\mathcal{A}^2\mathcal{F}+\mathcal{A}^3\mathcal{F}^2\Big)$$

$$+\frac{1}{15}\Big(\mathcal{A}\mathcal{F}\mathcal{A}^4+\mathcal{A}^3\mathcal{F}\mathcal{A}^2+\mathcal{A}^5\mathcal{F}\Big)-\frac{1}{35}\mathcal{A}^7\Big]\Big\}$$

$$\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) = n\int_0^1 dt\, t^{n-1}\mathrm{Tr}\left\{\mathcal{A}\Big[\mathcal{F}+(t-1)\mathcal{A}^2\Big]^{n-1}\right\}$$

To compute the **anomaly**, we use the **Russian formula**

$$\omega^0_{2n-1}(\mathcal{A},\mathcal{F}) = n\int_0^1 dt\, t^{n-1}\mathrm{Tr}\left\{\big(\mathcal{A}+v\big)\Big[\mathcal{F}+(t-1)\big(\mathcal{A}+v\big)^2\Big]^{n-1}\right\}$$

and expand to **first order** in ***v***

$$\boxed{\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) = n\int_0^1 dt\,(1-t)\mathrm{Tr}\Big[vd\Big(\mathcal{A}\mathcal{F}_t^{n-2}+\mathcal{F}_t\mathcal{A}\mathcal{F}_t^{n-3}+\ldots+\mathcal{F}_t^{n-1}\mathcal{A}\Big)\Big]}$$

or introducing the **symmetrized trace**

$$\omega^1_{2n-2}(v,\mathcal{A},\mathcal{F}) = n(n-1)\int_0^1 dt\,\mathrm{Str}\Big[vd\Big(\mathcal{F}_t^{n-2}\mathcal{A}\Big)\Big]$$

Let us apply this to recover the **four-dimensional case** (n = 3)

$$\omega_4^1(v,\mathcal{A},\mathcal{F}) = 3\int_0^1 dt\,(1-t)\mathrm{Tr}\left[vd\Big(\mathcal{A}\mathcal{F}_t+\mathcal{F}_t\mathcal{A}\Big)\right]$$

$$= 3\int_0^1 dt\,(1-t)\mathrm{Tr}\left\{vd\Big[t\mathcal{A}\mathcal{F}+t\mathcal{F}\mathcal{A}+2t(t-1)\mathcal{A}^3\Big]\right\}$$

$$= \frac{1}{2}\mathrm{Tr}\left[vd\Big(\mathcal{A}\mathcal{F}+\mathcal{F}\mathcal{A}-\mathcal{A}^3\Big)\right] = \frac{1}{2}\mathrm{Tr}\left[vd\Big(\mathcal{A}d\mathcal{A}+d\mathcal{A}\mathcal{A}+\mathcal{A}^3\Big)\right]$$

$$= \mathrm{Tr}\left[vd\left(\mathcal{A}d\mathcal{A}+\frac{1}{2}\mathcal{A}^3\right)\right]$$

$$\int_{S^4}\mathscr{G}^1[v,\mathcal{A}] = \frac{i}{24\pi^2}\int_{S^4}\mathrm{Tr}\left[vd\left(\mathcal{A}d\mathcal{A}+\frac{1}{2}\mathcal{A}^2\right)\right]$$

while in six dimensions (n = 4) we find

$$\omega_6^1(v,\mathcal{A},\mathcal{F}) = 4\int_0^1 dt\,(1-t)\mathrm{Tr}\left[vd\Big(\mathcal{A}\mathcal{F}_t^2+\mathcal{F}_t\mathcal{A}\mathcal{F}_t+\mathcal{F}_t^2\mathcal{A}\Big)\right]$$

$$= 4\int_0^1 dt\,(1-t)\mathrm{Tr}\Big\{vd\Big[t^2\Big(\mathcal{A}\mathcal{F}^2+\mathcal{F}\mathcal{A}\mathcal{F}+\mathcal{F}^2\mathcal{A}\Big)$$

$$+t^2(t-1)\Big(2\mathcal{A}^3\mathcal{F}+\mathcal{A}\mathcal{F}\mathcal{A}^2+\mathcal{A}^2\mathcal{F}\mathcal{A}+2\mathcal{F}\mathcal{A}^3\Big)+3t^2(t-1)^2\mathcal{A}^5\Big]\Big\}$$

$$= 4\int_0^1 dt\,(1-t)\mathrm{Tr}\Big\{vd\Big[t^2\Big(\mathcal{A}\mathcal{F}^2+\mathcal{F}\mathcal{A}\mathcal{F}+\mathcal{F}^2\mathcal{A}\Big)$$

$$+t^2(t-1)\Big(2\mathcal{A}^3\mathcal{F}+\mathcal{A}\mathcal{F}\mathcal{A}^2+\mathcal{A}^2\mathcal{F}\mathcal{A}+2\mathcal{F}\mathcal{A}^3\Big)+3t^2(t-1)^2\mathcal{A}^5\Big]\Big\}$$

$$= \frac{1}{3}\mathrm{Tr}\Big\{vd\Big[\Big(\mathcal{A}\mathcal{F}^2+\mathcal{F}\mathcal{A}\mathcal{F}+\mathcal{F}^2\mathcal{A}\Big)$$

$$-\frac{2}{5}\Big(2\mathcal{A}^3\mathcal{F}+\mathcal{A}\mathcal{F}\mathcal{A}^2+\mathcal{A}^2\mathcal{F}\mathcal{A}+2\mathcal{F}\mathcal{A}^3\Big)+\frac{3}{5}\mathcal{A}^5\Big]\Big\}$$

while in six dimensions (n = 4) we find

The gauge anomaly in **six dimensions**:

$$\omega_6^1(v,\mathcal{A}) = \mathrm{Tr}\left\{vd\left[\mathcal{A}(d\mathcal{A})^2 + \frac{1}{5}\Big(2\mathcal{A}^3 d\mathcal{A} + \mathcal{A}d\mathcal{A}\mathcal{A}^2 + \mathcal{A}^2 d\mathcal{A}\mathcal{A} + 2d\mathcal{A}\mathcal{A}^3\Big) + \frac{2}{5}\mathcal{A}^5\right]\right\}$$

$$\int_{S^6}\mathscr{G}^1[v,\mathcal{A}] = \int_{S^6}\mathrm{Tr}\left\{vd\left[\mathcal{A}(d\mathcal{A})^2 + \frac{1}{5}\Big(2\mathcal{A}^3 d\mathcal{A} + \mathcal{A}d\mathcal{A}\mathcal{A}^2 + \mathcal{A}^2 d\mathcal{A}\mathcal{A} + 2d\mathcal{A}\mathcal{A}^3\Big) + \frac{2}{5}\mathcal{A}^5\right]\right\}$$

$$+\, t^2(t-1)\Big(2\mathcal{A}^3\mathcal{F} + \mathcal{A}\mathcal{F}\mathcal{A}^2 + \mathcal{A}^2\mathcal{F}\mathcal{A} + 2\mathcal{F}\mathcal{A}^3\Big) + 3t^2(t-1)^2\mathcal{A}^5\Big]\Big\}$$

$$= \frac{1}{3}\mathrm{Tr}\left\{vd\Big[\Big(\mathcal{A}\mathcal{F}^2 + \mathcal{F}\mathcal{A}\mathcal{F} + \mathcal{F}^2\mathcal{A}\Big)\right.$$

$$\left. -\frac{2}{5}\Big(2\mathcal{A}^3\mathcal{F} + \mathcal{A}\mathcal{F}\mathcal{A}^2 + \mathcal{A}^2\mathcal{F}\mathcal{A} + 2\mathcal{F}\mathcal{A}^3\Big) + \frac{3}{5}\mathcal{A}^5\Big]\right\}$$

Consistent vs. covariant anomalies

So far, we have dealt with the so-called **consistent anomaly** derived from the anomalous action functional and satisfying the **Wess-Zumino consistency condition**

$$\Gamma[\mathcal{A}] = c_n \int_{D_{2n-1}} \omega^0_{2n-1}(\mathcal{A},\mathcal{F})$$

$$s\Gamma[\mathcal{A}] \equiv \int_{S^{2n-2}} \mathscr{G}^1[v,\mathcal{A}] = -c_n \int_{S^{2n-2}} \omega^1_{2n-2}(v,\mathcal{A},\mathcal{F})$$

The **(2n-3)-form dual consistent current** J is defined by

$$\delta_u\Gamma[\mathcal{A}] \equiv \int_{S^{2n-2}} \mathrm{Tr}\left[(Du)J\right] = -\int_{S^{2n-2}} \mathrm{Tr}\left(uDJ\right)$$

while the **1-form consistent current** j is given by

$$j = \star J$$

$$DJ = \mathscr{G}[\mathcal{A}] \quad \longrightarrow \quad (\star D \star) j = \star\mathscr{G}[\mathcal{A}]$$

The **consistent** form of the **anomaly** obtained

$$\mathrm{Tr}\left(uDJ\right)\equiv\mathscr{G}[u,\mathcal{A}]=-\frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\omega^1_{2n-2}(u,\mathcal{A},\mathcal{F})$$

is however **not gauge covariant**.

Question: Is there a **term** to be added to the consistent current

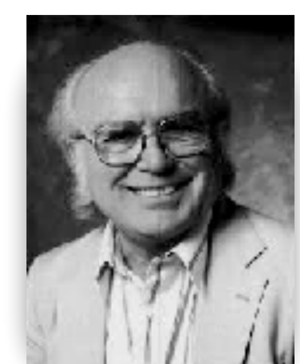
$$J_{\mathrm{cov}}\equiv J+J_{\mathrm{BZ}}$$

such that the **associated anomaly**

$$\mathrm{Tr}\left(uDJ_{\mathrm{cov}}\right)=\mathscr{G}[u,\mathcal{A}]+\mathrm{Tr}\left(uDJ_{\mathrm{BZ}}\right)\equiv\mathscr{G}[u,\mathcal{A}]_{\mathrm{cov}}$$

is **gauge covariant**?

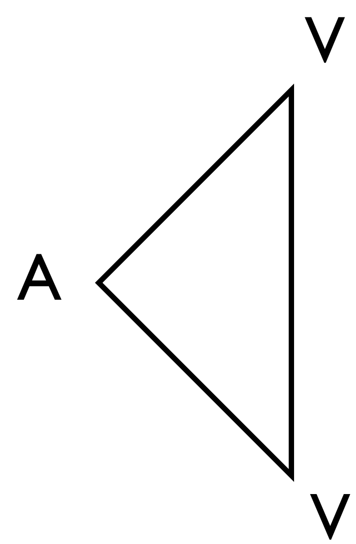
William A. Bardeen
(b. 1941)



Bruno Zumino
(1923-2014)

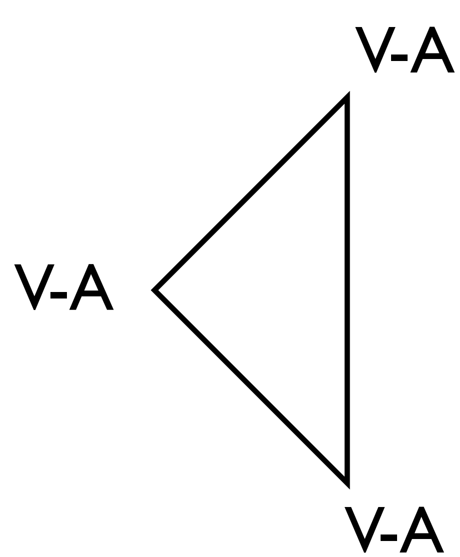
The question of **consistent vs. covariant** form of the anomaly **pops up** already **in perturbation theory**.

By computing a AVV triangle, we get a **covariant** result



$$\sim \epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left(T^a\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\right)$$

whereas the triangle with three V-A vertices renders a **noncovariant** anomaly

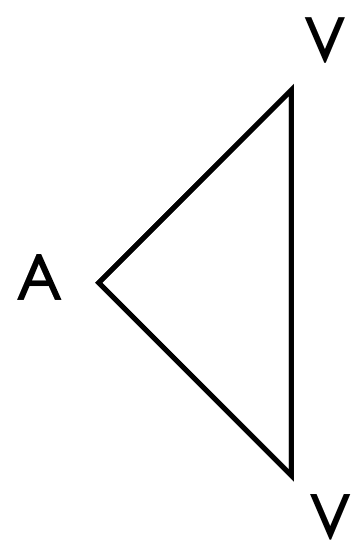


$$\sim \epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[T^a\partial_\mu\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta-\frac{i}{2}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right)\right]$$

The question of **consistent vs. covariant** form of the anomaly **pops up** already **in perturbation theory**.

By computing a AVV triangle, we get a **covariant** result

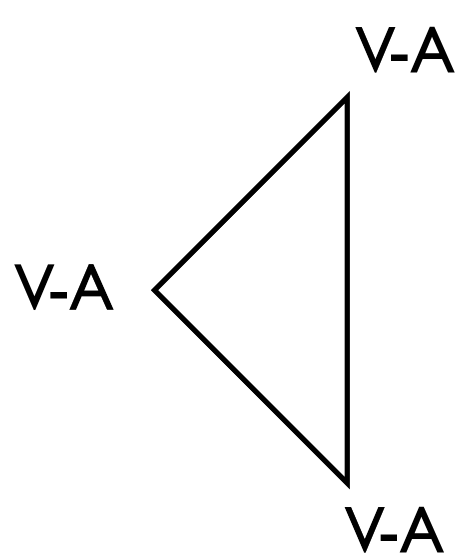
EASIER



$$\sim \epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left(T^a \mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\right)$$

whereas the triangle with three V-A vertices renders a **noncovariant** anomaly

HARDER



$$\sim \epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[T^a \partial_\mu\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta - \frac{i}{2}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right)\right]$$

Let us see how the Chern-Simons form transforms under **shifts in the connection**

$$\delta\mathcal{A} = \epsilon$$

Applying the **generalized transgression** (with $p = 0$) formula to

$$\mathcal{A}_t = \mathcal{A} + t\epsilon \qquad\qquad \mathscr{Q} = \omega^0_{2n-1}(\mathcal{A}, \mathcal{F})$$

we find

$$\begin{aligned}\omega^0_{2n-1}(\mathcal{A}+\epsilon) - \omega^0_{2n-1}(\mathcal{A}) &= \int_0^1 \ell_t d\omega^0_{2n-1}(\mathcal{A}_t, \mathcal{F}_t) + d\int_0^1 \ell_t \omega^0_{2n-1}(\mathcal{A}_t, \mathcal{F}_t) \\ &= \int_0^1 \ell_t \mathrm{Tr}\,\mathcal{F}_t^n + d\int_0^1 \ell_t \omega^0_{2n-1}(\mathcal{A}_t, \mathcal{F}_t)\end{aligned}$$

with

$$\ell_t \mathcal{A}_t = 0$$

$$\ell_t \mathcal{F}_t = dt\,\epsilon$$

$$\delta_\epsilon \omega^0_{2n-1}(\mathcal{A},\mathcal{F}) = \int_0^1 \ell_t \operatorname{Tr}\mathcal{F}_t^n + d\int_0^1 \ell_t \omega^0_{2n-1}(\mathcal{A}_t,\mathcal{F}_t)$$

The integrand of the **first term** gives

$$\mathcal{A}_t = \mathcal{A} + t\epsilon$$

$$\ell_t \operatorname{Tr}\mathcal{F}_t^n = n\operatorname{Tr}\left(\epsilon\mathcal{F}_t^{n-1}\right) = n\operatorname{Tr}\left(\epsilon\mathcal{F}^{n-1}\right) + \mathcal{O}(\epsilon^2)$$

For the **second one**

$$\begin{aligned}\ell_\epsilon\mathcal{F}_t &= \ell_\epsilon\Big[t\mathcal{F} + t(t-1)\mathcal{A}^2\Big] \\ &= tdt\epsilon\end{aligned}$$

$$\ell_t \omega^0_{2n-1}(\mathcal{A}_t,\mathcal{F}_t) = n\int_0^1 \ell_t \operatorname{Tr}\left(\mathcal{A}\mathcal{F}_t^{n-1}\right)$$

$$= \int_0^1 \operatorname{Tr}\Big(\mathcal{A}\ell_t\mathcal{F}_t\mathcal{F}_t^{n-2} + \mathcal{A}\mathcal{F}_t\ell_t\mathcal{F}_t\mathcal{F}_t^{n-3} + \ldots + \mathcal{A}\mathcal{F}_t^{n-2}\ell_t\mathcal{F}_t\Big)$$

$$= n\int_0^1 tdt\operatorname{Tr}\left[\epsilon\Big(\mathcal{F}_t^{n-2}\mathcal{A} + \mathcal{F}_t^{n-3}\mathcal{A}\mathcal{F}_t + \ldots + \mathcal{A}\mathcal{F}_t^{n-2}\Big)\right]$$

$$\boxed{c_n \equiv \frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}}$$

$$\delta_\epsilon \Gamma[\mathcal{A}] = c_n \int_{D_{2n-2}} \delta_\epsilon \omega^0_{2n-1}(\mathcal{A},\mathcal{F})$$

$$= c_n \int_{D_{2n-1}} \int_0^1 \ell_t \operatorname{Tr} \mathcal{F}_t^n + c_n \int_{S^{2n-2}} \int_0^1 \ell_t \omega^0_{2n-1}(\mathcal{A}_t,\mathcal{F}_t)$$

where we have computed

$$\ell_t \operatorname{Tr} \mathcal{F}_t^n = n \operatorname{Tr}\left(\epsilon \mathcal{F}^{n-1}\right)$$

$$\ell_t \omega^0_{2n-1}(\mathcal{A}_t,\mathcal{F}_t) = n \int_0^1 t dt \operatorname{Tr}\left[\epsilon\Big(\mathcal{F}_t^{n-2}\mathcal{A} + \mathcal{F}_t^{n-3}\mathcal{A}\mathcal{F}_t + \ldots + \mathcal{A}\mathcal{F}_t^{n-2}\Big)\right]$$

This gives the general structure

$$\delta_\epsilon \Gamma[\mathcal{A}] = \int_{D_{2n-1}} \operatorname{Tr}\left(\epsilon J_{\text{bdy}}\right) + \int_{S^{2n-2}} \operatorname{Tr}\left(\epsilon X\right)$$

$$\delta_\epsilon \Gamma[\mathcal{A}] = \int_{D_{2n-1}} \mathrm{Tr}\left(\epsilon J_{\mathrm{bulk}}\right) + \int_{S^{2n-2}} \mathrm{Tr}\left(\epsilon X\right)$$

with

$$J_{\mathrm{bulk}} = n c_n \mathcal{F}^{n-1}$$

$$X = n c_n \int_0^1 t dt \left(\mathcal{F}_t^{n-2}\mathcal{A} + \mathcal{F}_t^{n-3}\mathcal{A}\mathcal{F}_t + \ldots + \mathcal{A}\mathcal{F}_t^{n-2}\right)$$

If we **particularize** this to the case of a **gauge transformation**

$$\epsilon = Du$$

$$\delta_u \Gamma[\mathcal{A}] = \int_{D_{2n-1}} \mathrm{Tr}\left[(Du) J_{\mathrm{bulk}}\right] + \int_{S^{2n-2}} \mathrm{Tr}\left[(Du)X\right]$$

Stokes theorem

$$= -\int_{D_{2n-1}} \mathrm{Tr}\left(u D J_{\mathrm{bulk}}\right) + \int_{S^{2n-2}} \mathrm{Tr}\left[u\left(J_{\mathrm{bulk}} - DX\right)\right]$$

$$\delta_u\Gamma[\mathcal{A}] = -\int_{D_{2n-1}} \mathrm{Tr}\left(uDJ_{\mathrm{bulk}}\right) + \int_{S^{2n-2}} \mathrm{Tr}\left[u\big(J_{\mathrm{bulk}} - DX\big)\right]$$

But now, using the **Bianchi identity** $D\mathcal{F} = 0$

$$DJ_{\mathrm{bulk}} = nc_n D\Big(\mathrm{Tr}\,\mathcal{F}^{n-1}\Big) = 0$$

so the gauge variation of the effective action is **local**

$$\delta_u\Gamma[\mathcal{A}] = \int_{S^{2n-2}} \mathrm{Tr}\left[u\big(J_{\mathrm{bulk}} - DX\big)\right] \equiv -\int_{S^{2n-2}} \mathrm{Tr}\left(u\mathscr{G}[\mathcal{A}]_{\mathrm{cons}}\right)$$

$$\int_{S^{2n-2}} \mathrm{Tr}\left(uDJ\right) = \int_{S^{2n-2}} \mathrm{Tr}\left(u\mathscr{G}[\mathcal{A}]_{\mathrm{cons}}\right)$$

$$D(J-X) = -J_{\mathrm{bulk}}\Big|_{S^{2n-2}} = -\frac{1}{(n-1)!}\frac{i^n}{(2\pi)^{n-1}}\mathrm{Tr}\,\mathcal{F}^{n-1}$$

$$D(J-X) = J_{\text{bulk}}\Big|_{S^{2n-2}} = \frac{1}{(n-1)!}\frac{i^n}{(2\pi)^{n-1}}\text{Tr}\,\mathcal{F}^{n-1}$$

With this, we identify the **Bardeen-Zumino term** $\boldsymbol{J_{\text{BZ}} = -X}$

$$\boxed{J_{\text{BZ}} = -\frac{1}{(n-1)!}\frac{i^n}{(2\pi)^{n-1}}\int_0^1 tdt\left(\mathcal{F}_t^{n-2}\mathcal{A} + \mathcal{F}_t^{n-3}\mathcal{A}\mathcal{F}_t + \ldots + \mathcal{A}\mathcal{F}_t^{n-2}\right)}$$

The covariant current is then given by

$$J_{\text{cov}} \equiv J + J_{\text{BZ}}$$

whose divergence gives the **covariant anomaly**

$$\boxed{\text{Tr}\left(uDJ_{\text{cov}}\right) \equiv \mathscr{G}[u,\mathcal{A}]_{\text{cov}} = -\frac{1}{(n-1)!}\frac{i^n}{(2\pi)^{n-1}}\text{Tr}\left(u\mathcal{F}^{n-1}\right)}$$

Let us find the **Bardeen-Zumino term** and the **covariant anomaly** in **four dimensions** (n = 3)

$$J_{\rm BZ} = \frac{i}{8\pi^2}\int_0^1 tdt\left(\mathcal{F}_t\mathcal{A}+\mathcal{A}\mathcal{F}_t\right)$$

$$= \frac{i}{8\pi^2}\int_0^1 t^2dt\Big[\mathcal{F}\mathcal{A}+\mathcal{A}\mathcal{F}+2(t-1)\mathcal{A}^3\Big]$$

$$= \frac{i}{24\pi^2}\left(\mathcal{F}\mathcal{A}+\mathcal{A}\mathcal{F}-\frac{1}{2}\mathcal{A}^3\right)$$

The **four-dimensional covariant anomaly** is then given by

$$\mathscr{G}_a[\mathcal{A}]_{\rm cov} = \frac{i}{8\pi^2}\mathrm{Tr}\left(T^a\mathcal{F}^2\right)$$

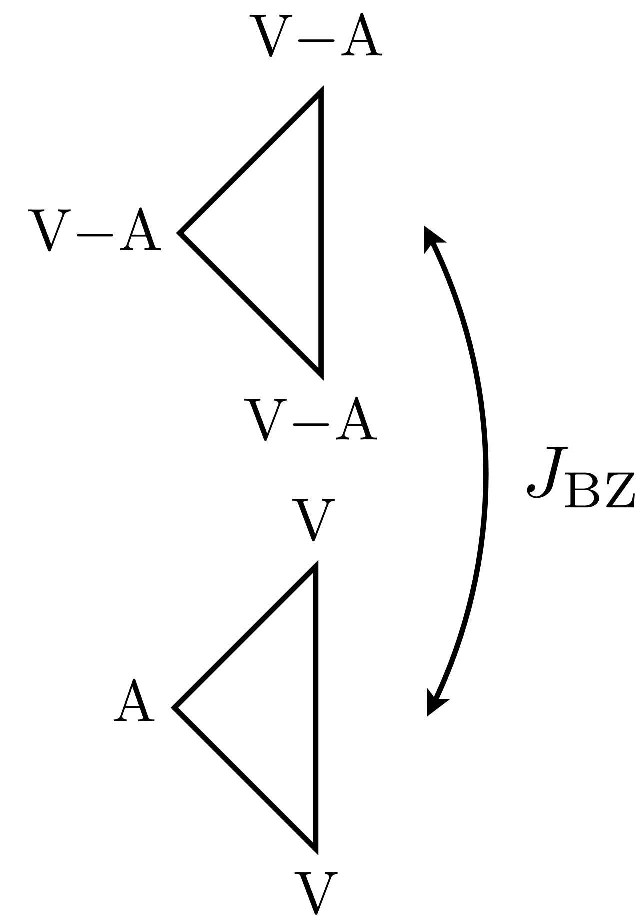
back to Minkowski signature

$$\mathscr{G}_a[\mathcal{A}]_{\rm cov} = \frac{1}{8\pi^2}\mathrm{Tr}\left(T^a\mathcal{F}^2\right)$$

Thus, in **four dimensions** the expressions of the **consistent** and **covariant** forms of the anomaly are

$$\mathscr{G}_a[\mathcal{A}]_{\rm cons} = \frac{1}{24\pi^2}\mathrm{Tr}\,\left[T^a d\left(\mathcal{A}d\mathcal{A} + \frac{1}{2}\mathcal{A}^3\right)\right]$$

$$\mathscr{G}_a[\mathcal{A}]_{\rm cov} = \frac{1}{8\pi^2}\mathrm{Tr}\,\left(T^a\mathcal{F}^2\right)$$



In the **Abelian case**, the difference between the **covariant** and the **consistent anomaly** is in the **prefactor**

$$\mathscr{G}_a[\mathcal{A}]_{\rm cons} = \frac{1}{3}\mathscr{G}_a[\mathcal{A}]_{\rm cov} = \frac{1}{24\pi^2}\mathrm{Tr}\,\left(T^a\mathcal{F}^2\right)$$

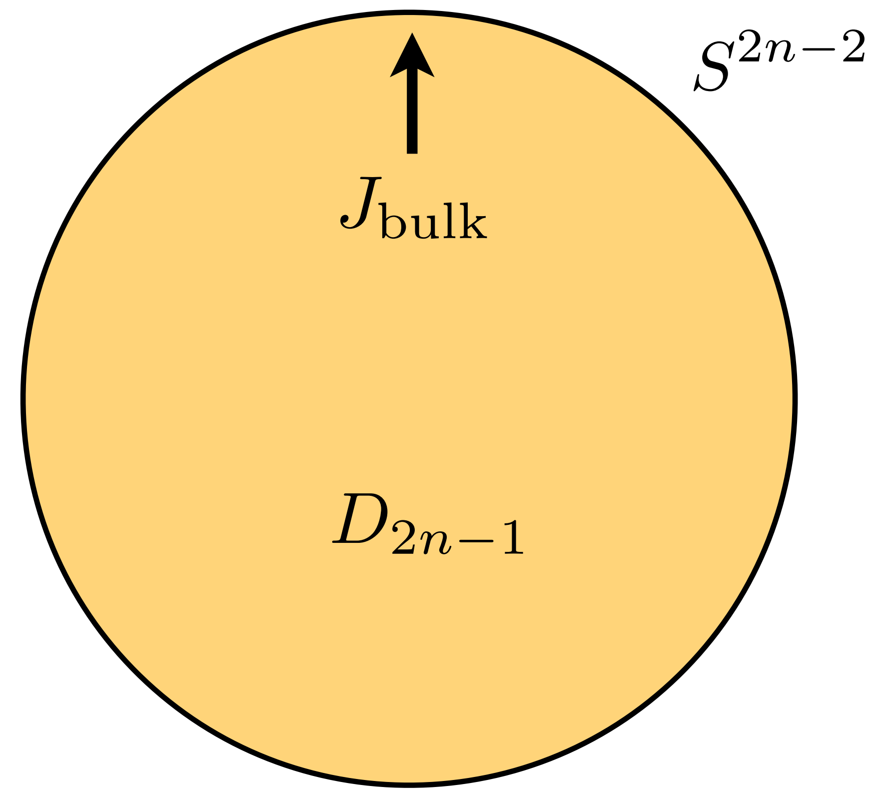
Physical interpretation: anomaly inflow

We have found that the **(integrated) covariant anomaly** is given by the **flux** of the **bulk current** over $S^{2n-2} = \partial D_{2n-1}$

$$\int_{S^{2n-2}} \mathrm{Tr}\,(uJ_{\mathrm{bulk}}) = -\int_{S^{2n-2}} \mathrm{Tr}\left(u\mathscr{G}[\mathcal{A}]_{\mathrm{cov}}\right)$$

The bulk current is **anomaly-free** in D_{2n-1}

$$DJ_{\mathrm{bulk}} = 0$$



The **charge flow** from the bulk into the boundary renders the **gauge theory on S^{2n-2}** anomalous

This **flow** is controlled by the **covariant anomaly**

Revisiting the Bardeen anomaly (without Feynman diagrams)

Let us consider again the theory of a **left-handed** and a **right-handed** fermion coupled respectively to **gauge fields** $\mathcal{A}_L$ and $\mathcal{A}_R$


Juan L. Mañes
(b. 1955)

Naively, we would start with the **anomaly polynomial**

$$\mathrm{Tr}\,\mathcal{F}_L^n - \mathrm{Tr}\,\mathcal{F}_R^n = d\Big[\omega_{2n-1}^0(\mathcal{A}_L,\mathcal{F}_L) - \omega_{2n-1}^0(\mathcal{A}_R,\mathcal{F}_R)\Big]$$

leading to the **anomalous effective action**

$$\Gamma[\mathcal{A}_L,\mathcal{A}_R] = c_n \int_{D_{2n-1}} \Big[\omega_{2n-1}^0(\mathcal{A}_L,\mathcal{F}_L) - \omega_{2n-1}^0(\mathcal{A}_R,\mathcal{F}_R)\Big]$$

This action, however, **transforms** under **vector gauge transformations**

$$\delta_V\mathcal{A}_L = \delta_V\mathcal{A}_R = Du$$

$$\Downarrow$$

$$\delta_V\Gamma[\mathcal{A}_L,\mathcal{A}_R] = c_n \int_{S^{2n-2}} \Big[\omega_{2n-1}^1(u,\mathcal{A}_L,\mathcal{F}_L) - \omega_{2n-1}^1(u,\mathcal{A}_R,\mathcal{F}_R)\Big] \neq 0$$

To make the theory **invariant** under **vector gauge transformations**, we use the freedom to add a **local counterterm**

$$\Gamma[\mathcal{A}_L, \mathcal{A}_R] = c_n \int_{D_{2n-1}} \Big[\omega^0_{2n-1}(\mathcal{A}_L, \mathcal{F}_L) - \omega^0_{2n-1}(\mathcal{A}_R, \mathcal{F}_R)$$

Bardeen counterterm

$$+ dS_{2n-2}(\mathcal{A}_L, \mathcal{A}_R, \mathcal{F}_L, \mathcal{F}_R)\Big]$$

To compute this counterterm, we introduce the **family** of connections interpolating between $\mathcal{A}_L$ and $\mathcal{A}_R$

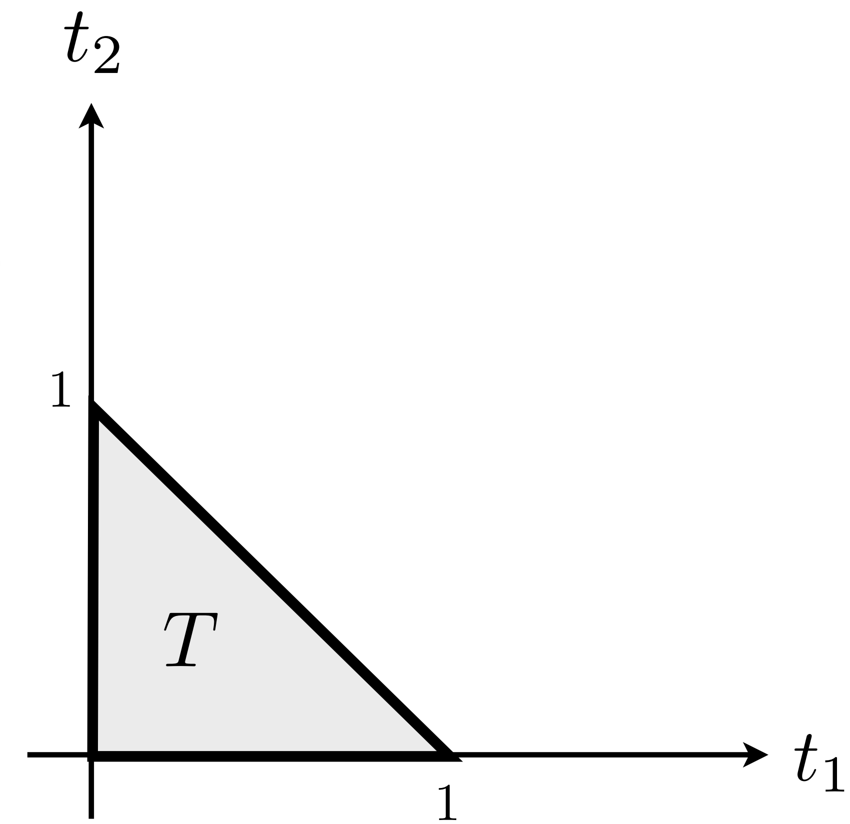
$$\mathcal{A}_{t_1 t_2} = t_1 \mathcal{A}_L + t_2 \mathcal{A}_R \qquad \text{with} \qquad 0 \leq t_1, t_2 \leq 1$$

$$\begin{aligned}\mathcal{F}_{t_1 t_2} &= d\mathcal{A}_{t_1 t_2} + \mathcal{A}^2_{t_1 t_2} \\ &= t_1 \mathcal{F}_L + t_2 \mathcal{F}_R + t_1(t_1 - 1)\mathcal{A}^2_L + (t_2 - 1)\mathcal{A}^2_R + t_1 t_2\Big(\mathcal{A}_L \mathcal{A}_R + \mathcal{A}_R \mathcal{A}_L\Big)\end{aligned}$$

We apply now the **generalized transgression** formula with $p=1$ to

$$\mathscr{Q} = \mathrm{Tr}\,\mathcal{F}^n_{t_1 t_2} \qquad\qquad (q=0)$$

and take the domain T to be the triangle



$$\int_{\partial T} \ell_t \mathrm{Tr}\,\mathcal{F}^n_{t_1 t_2} = \frac{1}{2}\int_T \ell_t^2 d\big(\mathrm{Tr}\,\mathcal{F}^n_{t_1 t_2}\big) - \frac{1}{2} d\int_T \ell_t^2 \mathrm{Tr}\,\mathcal{F}^n_{t_1 t_2}$$

Using

$$\ell_t \mathcal{A}_{t_1 t_2} = 0$$

$$\ell_t \mathcal{F}_{t_1 t_2} = d_t \mathcal{A}_{t_1 t_2} = \left(dt_1 \frac{\partial}{\partial t_1} + dt_2 \frac{\partial}{\partial t_2}\right) \mathcal{A}_{t_1 t_2} = dt_1 \mathcal{A}_L + dt_2 \mathcal{A}_R$$

we compute each term of the transgression formula

$$\int_{\partial T} \ell_t \mathrm{Tr}\,\mathcal{F}_{t_1 t_2}^n = \frac{1}{2}\int_T \ell_t^2 d\big(\mathrm{Tr}\,\mathcal{F}_{t_1t_2}^n\big) - \frac{1}{2} d\int_T \ell_t^2 \mathrm{Tr}\,\mathcal{F}_{t_1t_2}^n$$

$$d\mathrm{Tr}\,\mathcal{F}_{t_1t_2}^n = 0$$

$$\ell_t \mathrm{Tr}\,\mathcal{F}_{t_1t_2}^n = n\mathrm{Tr}\left[\big(dt_1\mathcal{A}_L + dt_2\mathcal{A}_R\big)\mathcal{F}_{t_1t_2}^{n-1}\right]$$

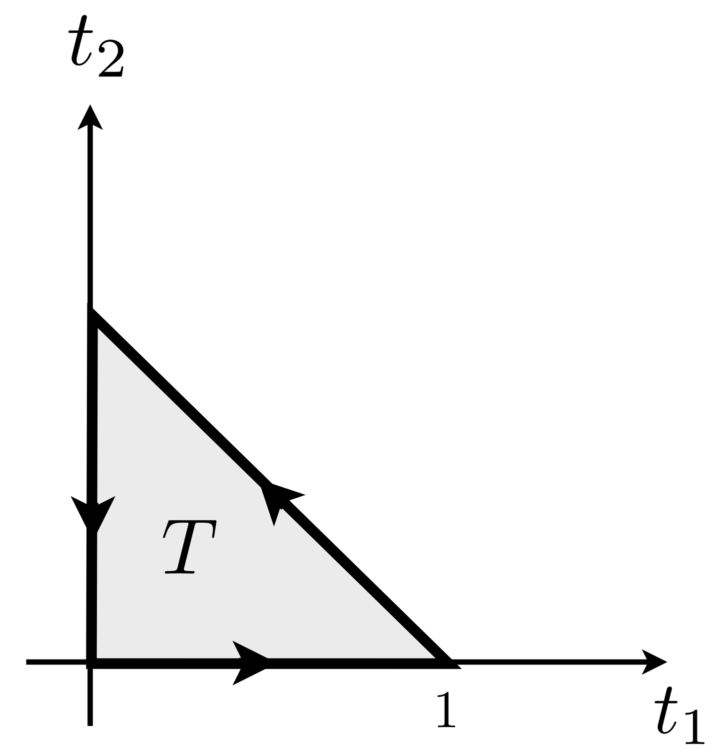
As for the **second term** on the **right-hand side**

$$\begin{aligned}
\ell_t^2 \mathrm{Tr}\,\mathcal{F}_{t_1t_2}^n &= n\ell_t \mathrm{Tr}\left(d_t\mathcal{A}_{t_1t_2}\mathcal{F}_{t_1t_2}^{n-1}\right) = n(n-1)\mathrm{Tr}\left(d_t\mathcal{A}_{t_1t_2}d_t\mathcal{A}_{t_1t_2}\mathcal{F}_{t_1t_2}^{n-2}\right) \\
&= n(n-1)\mathrm{Tr}\left[\big(dt_1\mathcal{A}_L + dt_2\mathcal{A}_R\big)\big(dt_1\mathcal{A}_L + dt_2\mathcal{A}_R\big)\mathcal{F}_{t_1t_2}^{n-2}\right] \\
&= n(n-1)dt_1dt_2\mathrm{Tr}\left[\big(\mathcal{A}_L\mathcal{A}_R - \mathcal{A}_R\mathcal{A}_L\big)\mathcal{F}_{t_1t_2}^{n-2}\right] \\
&= n(n-1)d^2t\mathrm{Tr}\left[\big(\mathcal{A}_L\mathcal{A}_R - \mathcal{A}_R\mathcal{A}_L\big)\mathcal{F}_{t_1t_2}^{n-2}\right]
\end{aligned}$$

$$n\int_{\partial T}\mathrm{Tr}\left[\big(dt_1\mathcal{A}_L + dt_2\mathcal{A}_R\big)\mathcal{F}_{t_1t_2}^{n-1}\right] = -\frac{1}{2}n(n-1)d\int_T d^2t\,\mathrm{Tr}\left[\big(\mathcal{A}_L\mathcal{A}_R - \mathcal{A}_R\mathcal{A}_L\big)\mathcal{F}_{t_1t_2}^{n-2}\right]$$

$$n\int_{\partial T}\mathrm{Tr}\left[(dt_1\mathcal{A}_L+dt_2\mathcal{A}_R)\mathcal{F}_{t_1t_2}^{n-1}\right]=-\frac{1}{2}n(n-1)d\int_T d^2t\,\mathrm{Tr}\left[(\mathcal{A}_L\mathcal{A}_R-\mathcal{A}_R\mathcal{A}_L)\mathcal{F}_{t_1t_2}^{n-2}\right]$$

We compute the **left-hand** side:



$$n\int_{\partial T}\mathrm{Tr}\left[(dt_1\mathcal{A}_L+dt_2\mathcal{A}_R)\mathcal{F}_{t_1t_2}^{n-1}\right]=-n\int_0^1 dt\,\mathrm{Tr}\left(\mathcal{A}_R\mathcal{F}_{0,1-t}^{n-1}\right)$$

$$+n\int_0^1 dt\,\mathrm{Tr}\left(\mathcal{A}_L\mathcal{F}_{t,0}^{n-1}\right)$$

$$+n\int_0^1 dt\,\mathrm{Tr}\left[(\mathcal{A}_L-\mathcal{A}_R)\mathcal{F}_{t,1-t}^{n-1}\right]$$

$$\boxed{\begin{array}{c}\mathcal{F}_{0,1-t}=\mathcal{F}_{R,1-t}\\ \mathcal{F}_{t,0}=\mathcal{F}_{L,t}\end{array}}$$

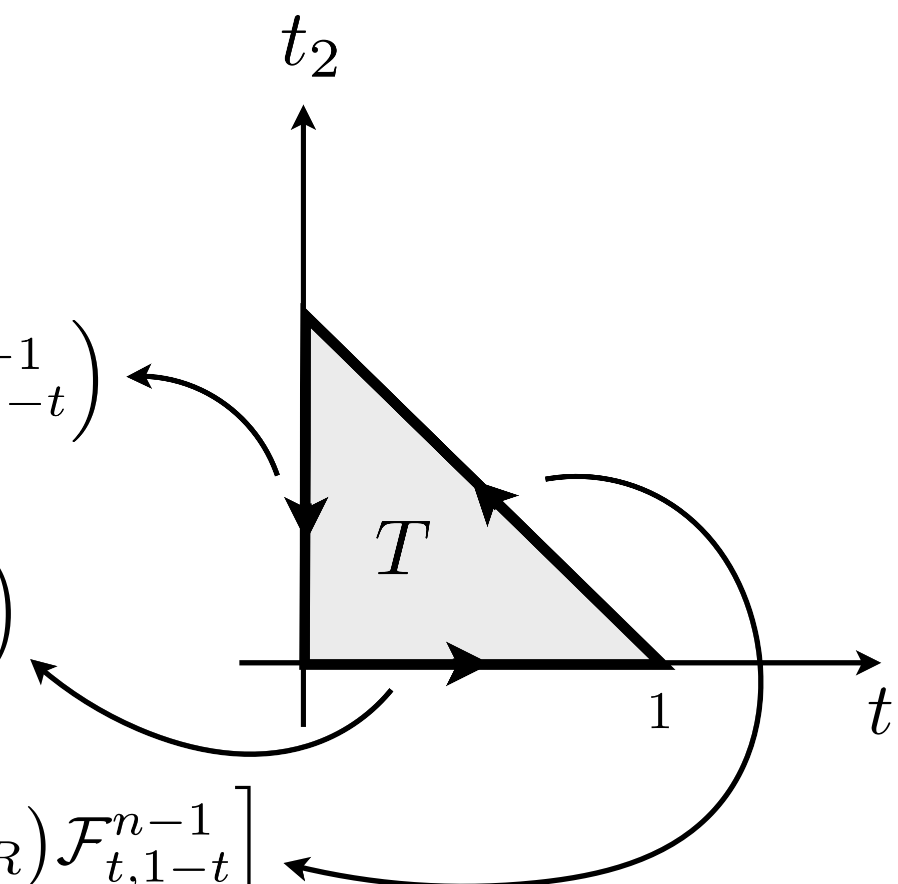
$$n\int_{\partial T}\mathrm{Tr}\left[(dt_1\mathcal{A}_L+dt_2\mathcal{A}_R)\mathcal{F}_{t_1t_2}^{n-1}\right]=\omega_{2n-1}^0(\mathcal{A}_L,\mathcal{F}_L)-\omega_{2n-1}^0(\mathcal{A}_R,\mathcal{F}_R)$$

$$+n\int_0^1 dt\,\mathrm{Tr}\left[(\mathcal{A}_L-\mathcal{A}_R)\mathcal{F}_{t,1-t}^{n-1}\right]$$

$$n\int_{\partial T}\mathrm{Tr}\Big[(dt_1\mathcal{A}_L+dt_2\mathcal{A}_R)\mathcal{F}_{t_1t_2}^{n-1}\Big]=-\frac{1}{2}n(n-1)d\int_T d^2t\,\mathrm{Tr}\Big[(\mathcal{A}_L\mathcal{A}_R-\mathcal{A}_R\mathcal{A}_L)\mathcal{F}_{t_1t_2}^{n-2}\Big]$$

We compute the **left-hand side**:

$$\begin{aligned}n\int_{\partial T}\mathrm{Tr}\Big[(dt_1\mathcal{A}_L+dt_2\mathcal{A}_R)\mathcal{F}_{t_1t_2}^{n-1}\Big]&=-n\int_0^1 dt\,\mathrm{Tr}\Big(\mathcal{A}_R\mathcal{F}_{0,1-t}^{n-1}\Big)\\&\quad+n\int_0^1 dt\,\mathrm{Tr}\Big(\mathcal{A}_L\mathcal{F}_{t,0}^{n-1}\Big)\\&\quad+n\int_0^1 dt\,\mathrm{Tr}\Big[(\mathcal{A}_L-\mathcal{A}_R)\mathcal{F}_{t,1-t}^{n-1}\Big]\end{aligned}$$



$$\boxed{\begin{aligned}\mathcal{F}_{0,1-t}&=\mathcal{F}_{R,1-t}\\ \mathcal{F}_{t,0}&=\mathcal{F}_{L,t}\end{aligned}}$$

$$n\int_{\partial T}\mathrm{Tr}\Big[(dt_1\mathcal{A}_L+dt_2\mathcal{A}_R)\mathcal{F}_{t_1t_2}^{n-1}\Big]=\omega_{2n-1}^0(\mathcal{A}_L,\mathcal{F}_L)-\omega_{2n-1}^0(\mathcal{A}_R,\mathcal{F}_R)$$

$$+n\int_0^1 dt\,\mathrm{Tr}\Big[(\mathcal{A}_L-\mathcal{A}_R)\mathcal{F}_{t,1-t}^{n-1}\Big]$$

$$n \int_{\partial T} \mathrm{Tr}\left[\big(dt_1 \mathcal{A}_L + dt_2 \mathcal{A}_R\big)\mathcal{F}_{t_1 t_2}^{n-1}\right] = -\frac{1}{2} n(n-1) d \int_T d^2 t\, \mathrm{Tr}\left[\big(\mathcal{A}_L \mathcal{A}_R - \mathcal{A}_R \mathcal{A}_L\big)\mathcal{F}_{t_1 t_2}^{n-2}\right]$$

$$n \int_{\partial T} \mathrm{Tr}\left[\big(dt_1 \mathcal{A}_L + dt_2 \mathcal{A}_R\big)\mathcal{F}_{t_1 t_2}^{n-1}\right] = \omega_{2n-1}^{0}(\mathcal{A}_L, \mathcal{F}_L) - \omega_{2n-1}^{0}(\mathcal{A}_R, \mathcal{F}_R)$$

$$+ n \int_0^1 dt\, \mathrm{Tr}\left[\big(\mathcal{A}_L - \mathcal{A}_R\big)\mathcal{F}_{t,1-t}^{n-1}\right]$$

Invariant under vector gauge transformations

$$\mathcal{F}_{t,1-t} = t\mathcal{F}_L + (1-t)\mathcal{F}_R + t(t-1)\big(\mathcal{A}_R - \mathcal{A}_L\big)^2$$

$$n \int_0^1 dt\, \mathrm{Tr}\left[\big(\mathcal{A}_R - \mathcal{A}_L\big)\mathcal{F}_{t,1-t}^{n-1}\right] = \omega_{2n-1}^{0}(\mathcal{A}_L, \mathcal{F}_L) - \omega_{2n-1}^{0}(\mathcal{A}_R, \mathcal{F}_R)$$

$$- \frac{1}{2} n(n-1) d \int_T d^2 t\, \mathrm{Tr}\left[\big(\mathcal{A}_L \mathcal{A}_R - \mathcal{A}_R \mathcal{A}_L\big)\mathcal{F}_{t_1 t_2}^{n-2}\right]$$

$$\widetilde{\omega}_{2n-1}^{0}(\mathcal{A}_L, \mathcal{A}_R, \mathcal{F}_L, \mathcal{F}_R)$$

$$\left(d\widetilde{\omega}_{2n-1}^{0} = \mathrm{Tr}\,\mathcal{F}_L^n - \mathrm{Tr}\,\mathcal{F}_R^n\right)$$

Bardeen counterterm

$$\widetilde{\omega}^0_{2n-1}(\mathcal{A}_L,\mathcal{A}_R,\mathcal{F}_L,\mathcal{F}_R)=\omega^0_{2n-1}(\mathcal{A}_L,\mathcal{F}_L)-\omega^0_{2n-1}(\mathcal{A}_R,\mathcal{F}_R)$$
$$-\frac{1}{2}n(n-1)d\int_T d^2t\,\mathrm{Tr}\left[(\mathcal{A}_L\mathcal{A}_R-\mathcal{A}_R\mathcal{A}_L)\mathcal{F}^{n-2}_{t_1 t_2}\right]$$
$$\mathcal{F}_{t,1-t}=t\mathcal{F}_L+(1-t)\mathcal{F}_R+t(t-1)\big(\mathcal{A}_R-\mathcal{A}_L\big)^2$$

It is easy to see that the Chern-Simons form

$$\widetilde{\omega}^0_{2n-1}(\mathcal{A}_L,\mathcal{A}_R,\mathcal{F}_L,\mathcal{F}_R)=n\int_0^1 dt\,\mathrm{Tr}\left[(\mathcal{A}_R-\mathcal{A}_L)\mathcal{F}^{n-1}_{t,1-t}\right]$$

not only reproduces the **appropriate anomaly polynomial**

$$d\widetilde{\omega}^0_{2n-1}(\mathcal{A}_L,\mathcal{A}_R,\mathcal{F}_L,\mathcal{F}_R)=\mathrm{Tr}\,\mathcal{F}^n_L-\mathrm{Tr}\,\mathcal{F}^n_R$$

but is also **invariant** under **vector gauge transformations**

$$\left.\begin{aligned}(\mathcal{A}_{L,R})_g&=g^{-1}\mathcal{A}_{L,R}g+g^{-1}dg\\(\mathcal{F}_{L,R})_g&=g^{-1}\mathcal{F}_{L,R}g\end{aligned}\right\}\Longrightarrow\quad\delta_V\widetilde{\omega}^0_{2n-1}(\mathcal{A}_L,\mathcal{A}_R,\mathcal{F}_L,\mathcal{F}_R)=0$$

$$\widetilde{\omega}^0_{2n-1}(\mathcal{A}_L,\mathcal{A}_R,\mathcal{F}_L,\mathcal{F}_R) = n\int_0^1 dt\,\mathrm{Tr}\left[(\mathcal{A}_R-\mathcal{A}_L)\mathcal{F}^{n-1}_{t,1-t}\right]$$

We compute the **effective action**

$$\Gamma[\mathcal{A}_L,\mathcal{A}_R] = \frac{1}{n!}\frac{i^n}{(2\pi)^{n-1}}\int_{D_{2n-1}}\widetilde{\omega}^0_{2n-1}(\mathcal{A}_L,\mathcal{A}_R,\mathcal{F}_L,\mathcal{F}_R)$$

To find the associated **anomaly**, we use the identity

$$\begin{aligned}\widetilde{\omega}^1_{2n-2}(u_{L,R},\mathcal{A}_{L,R},\mathcal{F}_{L,R}) &= \left(u_R\frac{\delta}{\delta\mathcal{A}_R}+u_L\frac{\delta}{\delta\mathcal{A}_L}\right)\widetilde{\omega}^0_{2n-1}(\mathcal{A}_{L,R},\mathcal{F}_{L,R}) \\ &= n\left(u_R\frac{\delta}{\delta\mathcal{A}_R}+u_L\frac{\delta}{\delta\mathcal{A}_L}\right)\int_0^1 dt\,\mathrm{Tr}\left[(\mathcal{A}_L-\mathcal{A}_R)\mathcal{F}^{n-1}_{t,1-t}\right]\end{aligned}$$

with $\mathcal{F}_{t,1-t} = t\mathcal{F}_L+(1-t)\mathcal{F}_R+t(t-1)\big(\mathcal{A}_R-\mathcal{A}_L\big)^2$

In **four dimensions** (n = 3)

$$\widetilde{\omega}_5^0 = 3\int_0^1 dt\,\mathrm{Tr}\left\{\big(\mathcal{A}_R - \mathcal{A}_L\big)\Big[t\mathcal{F} + (1-t)\mathcal{F}_R + t(t-1)\big(\mathcal{A}_R - \mathcal{A}_L\big)^2\Big]^2\right\}$$

$$= 6\int_0^1 dt\,\mathrm{Tr}\left\{\mathcal{A}\Big[(1-2t)\mathcal{F}_A + \mathcal{F}_V + 4t(t-1)\mathcal{A}\Big]^2\right\}$$

and integrating over the parameter t

$$\widetilde{\omega}_5^0(\mathcal{V},\mathcal{A},\mathcal{F}_V,\mathcal{F}_A) = 6\mathrm{Tr}\left(\mathcal{A}\mathcal{F}_V^2 + \frac{1}{3}\mathcal{A}\mathcal{F}_A^2 - \frac{4}{3}\mathcal{A}^3\mathcal{F}_V + \frac{8}{15}\mathcal{A}^3\right)$$

The **anomalous effective action** is therefore given by

$$\Gamma[\mathcal{V},\mathcal{A}] = -\frac{i}{4\pi^2}\int_{D_5}\mathrm{Tr}\left(\mathcal{A}\mathcal{F}_V^2 + \frac{1}{3}\mathcal{A}\mathcal{F}_A^2 - \frac{4}{3}\mathcal{A}^3\mathcal{F}_V + \frac{8}{15}\mathcal{A}^3\right)$$

which is manifestly **invariant** under **vector gauge transformations**.

To get the anomaly, we compute

$$\widetilde{\omega}_4^1(u_A,\mathcal{V},\mathcal{A}) = 6\int_0^1 dt\,\mathrm{Tr}\left(u_A\left\{\Big[(1-2t)\mathcal{F}_A+\mathcal{F}_V+4t(t-1)\mathcal{A}^2\Big]^2\right.\right.$$
$$+\,4t(t-1)\Big\{\mathcal{A},\Big[(1-2t)\mathcal{F}_A+\mathcal{F}_V+4t(t-1)\mathcal{A}^2\Big]\mathcal{A}\Big\}$$
$$\left.\left.+\,4t(t-1)\Big\{\mathcal{A},\mathcal{A}\Big[(1-2t)\mathcal{F}_A+\mathcal{F}_V+4t(t-1)\mathcal{A}^2\Big]\Big\}\right\}\right)$$

$$\widetilde{\omega}_4^1(u_A,\mathcal{V},\mathcal{A}) = 6\mathrm{Tr}\left\{u_A\left[\mathcal{F}_V^2+\frac{1}{3}\mathcal{F}_A^2-\frac{4}{3}\Big(\mathcal{A}^2\mathcal{F}_V+\mathcal{A}\mathcal{F}_V\mathcal{A}+\mathcal{F}_V\mathcal{A}^2\Big)+\frac{8}{3}\mathcal{A}^4\right]\right\}$$

from where we get the **Bardeen anomaly**

$$\delta_A\Gamma[\mathcal{V},\mathcal{A}] = -\frac{i}{4\pi^2}\int_{S^4}\mathrm{Tr}\left\{u_A\left[\mathcal{F}_V^2+\frac{1}{3}\mathcal{F}_A^2-\frac{4}{3}\Big(\mathcal{A}^2\mathcal{F}_V+\mathcal{A}\mathcal{F}_V\mathcal{A}+\mathcal{F}_V\mathcal{A}^2\Big)+\frac{8}{3}\mathcal{A}^4\right]\right\}$$

Thank you