Lattice Field Theory Fundamentals

A. Pilar Hernández

Departamento de Física Teórica and IFIC
Universidad de Valencia and CSIC
Apart. 22085, 46071 Valencia

Lectures delivered at the Les Houches École d'Été de Physique Théorique

OXFORD
UNIVERSITY PRESS

In memory of my collaborator and friend Jan Wennekers

Preface

This is the first time a Les Houches summer school is fully devoted to Lattice Field Theory (LFT). This is timely as the progress in the field has been spectacular in the last years.

In these lectures I will concentrate on the basics and I will deal mostly with the analytical formulation of LFT. The remaining lectures will discuss in detail the essential algorithmic aspects, as well as the modern perspectives.

There are many good introductory books on the subject (Creutz, 1983; Montvay and Münster, 1994; Smit, 2002; Rothe, 2005; DeGrand and DeTar, 2006; Gattringer and Lang, 2010). The goal of my lectures will be to provide a short summary of the more basic contents of those books.

Contents

1

On the need of a lattice formulation of Quantum Field Theory

There is firm experimental evidence that the laws of particle physics are accurately described by a Quantum Field Theory (QFT).

The experiments at LEP and at flavour factories of the last decades (Amsler *et al.*, 2008) have established the validity of the Standard Model (SM) up to a level of precision of 1% or better. The Standard Model is a renormalizable QFT with a simple Lagrangian that fits in a t-shirt (Fig. 1.1).

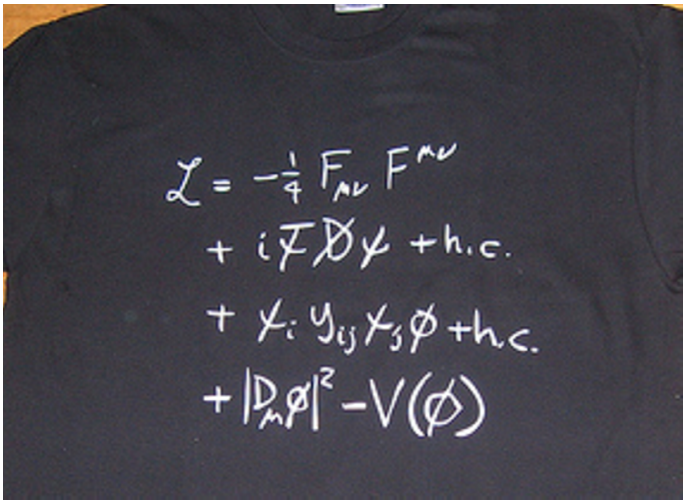


Fig. 1.1 t-shirt Standard Model

The pure gauge interactions depend on just three free parameters (the three coupling constants associated to the three gauge groups), and preserve the three discrete symmetries: parity (P), charge conjugation (C) and time-reversal (T). The matter-gauge interactions do not introduce any further free parameter in the model, but they violate P and C maximally, and preserve T. On the other hand, the interactions of the scalar Higgs field, that will be tested soon at the LHC, are poorly understood theoretically, as they bring real havoc to the theory. This sector contains many new free parameters that are required to fit data: 22-24 (depending on whether neutrinos are Majorana particles or not). It is responsible for the spontaneous breaking of the gauge symmetry, which is the fundamental pillar of the SM, and a mechanism we do not really understand. This sector also has the key to the subtle violation of CP and T symmetries in the SM.

Perturbation theory has given a great deal of information about the SM, but fails in various situations:

- In processes involving particles with $SU(3)$ interactions: quarks and gluons described by the beautifully simple QCD Lagrangian
$$\mathcal{L}_{QCD} = -\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_q \bar{\psi}_q (i\not{D} + m_q)\psi_q. \tag{1.1}$$
QCD is a strongly coupled theory at low energies, resulting in several phenomena that cannot be understood in perturbation theory: *confinement*, a *mass gap* and *spontaneous chiral symmetry breaking*. A precise quantitative understanding of QCD interactions is furthermore needed to test the quark-flavour sector of the SM, that is expected to be quite sensititive to new physics.
- The Higgs self-interactions are completely untested. The SM version of the Higgs potential:
$$V(\Phi) = -\frac{\mu^2}{2}\Phi^\dagger\Phi + \frac{\lambda}{4!}(\Phi^\dagger\Phi)^2, \tag{1.2}$$
is probably too naive. It suffers from the so-called *triviality problem*: the fact that the only renormalized value of the coupling is zero
$$\lim_{\Lambda\to\infty} \lambda_R = 0, \tag{1.3}$$
and is therefore a trivial theory, i.e. not phenomenologically viable. Only if there is a physical cutoff, the renormalized coupling can be non-vanishing, which would imply that the SM is an effective theory, valid only below some energy scale. If this is the case, however, the Higgs mass is expected to receive large quadratic corrections from the higher energy scales. This is the so-called hierarchy problem. Establishing the triviality of the SM Higgs potential requires to go beyond perturbation theory.
- Beyond the SM interactions (BSM): there are many alternatives to address the various open questions in the SM. Even though there is presently no compelling proposal to extend the SM at high energies (all alternatives involve more free parameters that the SM), it is quite likely that the SM is not the whole story. In many of the most popular theories BSM non-perturbative effects come into play: in SUSY non-perturbative effects are often invoked to break SUSY at low-energies, technicolor theories are up-scaled versions of QCD and the fashionable nearly conformal gauge theories also require a non-perturbative approach.
- Origin of chirality. The breaking of parity by the weak interactions is probably the most intriguing feature of the SM. It is notoriously difficult to ensure chiral gauge symmetry non-perturbatively. Finding such a formulation is likely to shed some light on the symmetry principles of the SM.

For all these reasons, having a non-perturbative tool to solve QFTs is essential. The only first-principles method is the regularization on a space-time grid that provides a non-perturbative definition of a regularized QFT (at least those that are asymptotically free), and can be treated in principle by numerical methods. Clearly this is not an easy task and the efforts of the lattice community in the past few decades were concentrated on Yang-Mills and QCD.

Solving Yang-Mills is not only the original goal of the lattice formulation, but it is still one of the famous Millennium Prize problems[1]. It will require the proof of the existence of Yang-Mills theory and the presence of a mass gap[2]. It would be great if any of the students in this school would solve this problem, becoming rich in more than one way...

In these lectures, I will review the foundations of the lattice formulation of scalar, fermion and gauge field theories, as well as QCD.

[1] http://www.claymath.org/millennium/Yang-Mills_Theory/yangmills.pdf

[2] *Prove that for any compact simple gauge group G, a non-trivial quantum YM theory exists in R^4 and has a non-vanishing mass gap (existence includes establishing axiomatic properties such as Osterwalder and Schrader)*

2
Basics of Quantum Field Theory

Quantum Field Theory is the synthesis of quantum mechanics and special relativity, which can be reached following two very different routes, as the relativistic limit of a system of many identical quantum particles and from the canonical quantization of classical fields.

2.1 Relativistic quantum mechanics and Fock space

The Hilbert space of a *fixed* number of quantum particles is not sufficient to describe the dynamics of a quantum system in the relativistic domain, because particles can be created/destroyed in collisions. The appropriate space to describe a relativistic quantum system is *Fock space*, the sum of all Hilbert spaces with any fixed number of particles:

$$\mathcal{F} = |0\rangle \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 ... \oplus \mathcal{H}_\infty, \tag{2.1}$$

where $|0\rangle$ is the vacuum state, which we assume normalized.

Owing to the (anti-)symmetrization properties of the physical states under permutation of identical particles, the states can be characterized by the occupation numbers, N_i, i.e. the number of particles in the energy-level E_i. It is possible to define creation and annihilation operators $\hat{a}_i^\dagger$ and $\hat{a}_i$ in Fock space that create/destroy one particle in the i-th level. If the particles are bosons, these operators have the following commutation relations:

$$\left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0, \quad \left[\hat{a}_i, \hat{a}_j\right] = 0, \quad \left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}. \tag{2.2}$$

We can use them to construct all the states of Fock space from the vacuum state $|0\rangle$:

$$|N_1, ..., N_n\rangle = \left(\hat{a}_1^\dagger\right)^{N_1} ... \left(\hat{a}_n^\dagger\right)^{N_n} |0\rangle. \tag{2.3}$$

An arbitrary observable is an operator in Fock space that can always be written in terms of creation/annihilation operators as:

$$\hat{O} = \sum_{n_1...n_n;m_1,...m_n} O_{n_1....n_n m_1...m_n} (\hat{a}_1)^{n_1} ... (\hat{a}_n)^{n_n} (\hat{a}_1^\dagger)^{m_1} ... (\hat{a}_n^\dagger)^{m_n}, \tag{2.4}$$

where $O_{n_1....m_n}$ are numbers such that $\hat{O}$ is Hermitian.

In the case of free particles, the one-particle states can be chosen to be momentum eigenstates and the spectrum is continuous:

$$|\mathbf{p}\rangle = \hat{a}^\dagger_{\mathbf{p}}|0\rangle, \quad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \tag{2.5}$$

The Lorentz-invariant normalization of these states is

$$\langle \mathbf{p}|\mathbf{p}'\rangle = (2\pi)^3 2E_{\mathbf{p}}\ \delta(\mathbf{p} - \mathbf{p}'). \tag{2.6}$$

We can define the so-called field operator:

$$\hat{\phi}^\dagger(\mathbf{x}) \equiv \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} e^{-i\mathbf{p}\cdot\mathbf{x}} \hat{a}^\dagger_{\mathbf{p}}, \tag{2.7}$$

which acts on the vacuum as

$$\hat{\phi}^\dagger(\mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} e^{-i\mathbf{p}\cdot\mathbf{x}} |\mathbf{p}\rangle \simeq |\mathbf{x}\rangle, \tag{2.8}$$

and can therefore be interpreted as creating a particle at point $\mathbf{x}$.

2.2 Canonical Field Quantization

We can also start with a real classical field $\phi(\mathbf{x}, t)$ with classical Lagrangian and Hamiltonian given by:

$$\mathcal{L}(\phi) = \int d^3x \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m_0^2\phi^2\right), \tag{2.9}$$

$$\mathcal{H}(\phi, \pi) = \int d^3x \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m_0^2\phi^2\right), \tag{2.10}$$

where

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \tag{2.11}$$

This classical system can be quantized canonically by identifying the pair of canonical variables, $\{\phi, \pi\}$, with quantum operators $\{\hat{\phi}, \hat{\pi}\}$.

In momentum space, it is easy to see that the Hamiltonian describes an infinite number of harmonic oscillators, one for each momentum, with one quantum of energy being

$$E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m_0^2}. \tag{2.12}$$

The well-known ladder operators $\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}$ of the harmonic oscillator are related to the quantum field operator by

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left\{\hat{a}_{\mathbf{p}} e^{-i(E_{\mathbf{p}}t - \mathbf{p}\mathbf{x})} + h.c.\right\}. \tag{2.13}$$

This operator resembles the time-evolved field operator in Fock space of eq. (2.7), if we identify the ladder operators of the harmonic oscillators with creation/annihilation operators in Fock space, providing therefore a particle interpretation to the quantized field.

Summarizing, the most important physical intuition is that a quantum field is a bunch of harmonic oscillators, whose ladder operators correspond to creation/annihilation operators in a Fock space. Particles are therefore interpreted as excitations of a quantum harmonic oscillator. This connection goes also in the opposite direction. Indeed, Weinberg has shown (Weinberg, 1995) that operators (observables) in Fock space, eq. (2.4), that satisfy the following conditions

Cluster decomposition	$\leftrightarrow$	locality
Hermiticity	$\leftrightarrow$	unitarity
Lorentz invariance	$\leftrightarrow$	causality

are necessarily functions of quantum field operators, such as eq. (2.13), resulting therefore in a Quantum Field Theory.

Continuous symmetries such as Lorentz invariance act in Fock space, according to Wigner's theorem, as unitary operators [1]. It is easy to see that the operator of space translations by the vector $\mathbf{x}$ is

$$\hat{U}_{\mathbf{x}} \equiv e^{-i\hat{\mathbf{P}}\mathbf{x}}, \tag{2.14}$$

where $\hat{\mathbf{P}}$ is the momentum operator.

The operator that implements time translations, by x_0, i.e. the quantum evolution operator is given in terms of the Hamiltonian by

$$\hat{U}_{x_0} \equiv e^{-i\hat{H}x_0}. \tag{2.15}$$

2.3 Field correlation functions and physical observables

The essential assumption that goes into the definition of cross sections and decay widths, is the existence of asymptotic states, which correspond to a bunch of non-interacting 1-particle states in the infinite past $t \to -\infty$ (in-states) as well as in the infinite future $t \to \infty$ (out-states). For this to happen two conditions are required:

- Localization of one-particle states or wave-packets
- Localization of the interaction: only when particles get sufficiently close are their interactions significant

[1] Some discrete symmetries such as time reversal are implemented by antiunitary operators, but we will not consider this case here.

The relation between the scattering matrix elements and field correlation functions is given by the LSZ (Lehmann, Symanzik and Zimmermann, 1955) reduction formula:

$$\prod_{i=1}^{n}\int d^4x_i e^{ip_i\cdot x_i}\prod_{j=1}^{k}\int d^4y_j e^{-iq_j\cdot y_i}\langle 0|T\left(\hat{\phi}(x_1)...\hat{\phi}(x_n)\hat{\phi}(y_1)....\hat{\phi}(y_k)\right)|0\rangle$$
$$\simeq_{p_i^0\to E_{\mathbf{p}_i},q_j^0\to E_{\mathbf{q}_j}}\prod_{i=1}^{n}\left(\frac{i\sqrt{Z}}{p_i^2-m^2+i\epsilon}\right)\prod_{j=1}^{k}\left(\frac{i\sqrt{Z}}{q_j^2-m^2+i\epsilon}\right)\langle \mathbf{p}_1,....,\mathbf{p}_n,out|\mathbf{q}_1,...,\mathbf{q}_k;in\rangle, \tag{2.16}$$

where the scattering amplitudes, $\langle \mathbf{p}_1,....,\mathbf{p}_n,out|\mathbf{q}_1,...,\mathbf{q}_k;in\rangle$, correspond to k asymptotic *in* states (one-particles states at $t=-\infty$) that end up as n *out* states (one-particle states at $t=\infty$). Z and m are the field renormalization constant and mass that characterize the asymptotic one-particle states. They can be extracted from the spectral representation of the two-point correlation functions, or Källen-Lehmann representation (Lehmann, 1954; Källen, 1972), that we derive in the next section. For a derivation of the LSZ formula see for example (Itzykson and Zuber, 1980; Peskin and Schroeder, 1995).

2.3.1 Källen-Lehmann Spectral representation of the propagator

The two-point correlation function or propagator is a fundamental quantity that allows to identify the asymptotic states, their masses and field renormalization factors. We review why this is so.

The invariance of the Hamiltonian under translations implies $\left[\hat{H},\hat{\mathbf{P}}\right]=0$, i.e. the eigenstates of the Hamiltonian are also momentum eigenstates;

$$\begin{aligned}\hat{\mathbf{P}}|\alpha(\mathbf{p})\rangle &= \mathbf{p}|\alpha(\mathbf{p})\rangle,\\ \hat{H}|\alpha(\mathbf{p})\rangle &= E_{\mathbf{p}}(\alpha)|\alpha(\mathbf{p})\rangle,\end{aligned} \tag{2.17}$$

with

$$E_{\mathbf{p}}^2(\alpha)=m(\alpha)^2+\mathbf{p}^2. \tag{2.18}$$

Here $m(\alpha)$ is not necessarily a one-particle mass, since $|\alpha(\mathbf{p})\rangle$ could represent a multiparticle state with total momentum $\mathbf{p}$, in which case it is simply the energy of the system in the rest frame, ie. where the total momentum vanishes. Therefore α labels all the energy eigenstates of zero momentum.

Using the completeness relation for the full Hilbert space:

$$\hat{1}=|0\rangle\langle 0|+\sum_{\alpha}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\alpha)}|\alpha(\mathbf{p})\rangle\langle\alpha(\mathbf{p})|. \tag{2.19}$$

we can write the propagator as

$$\langle 0|T\left(\hat{\phi}(x)\hat{\phi}(0)\right)|0\rangle\Big|_{x_0>0}=\langle 0|\hat{\phi}(x)\hat{1}\hat{\phi}(0)|0\rangle=\langle 0|\hat{\phi}(0)e^{-i\hat{P}\cdot x}\hat{1}\hat{\phi}(0)|0\rangle$$

$$= \sum_{\alpha} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\alpha)} \, e^{-ip\cdot x}\big|_{p_0=E_{\mathbf{p}}(\alpha)} \, |\langle 0|\hat{\phi}(0)|\alpha(\mathbf{p})\rangle|^2, \tag{2.20}$$

where we have used that the operator that implements space-time translations by x is $\hat{U}_x = e^{-i\hat{P}x}$, where $\hat{P}_0 = \hat{H}$, according to eqs. (2.14) and (2.15). We have furthermore assumed that the vacuum is invariant under temporal and spatial translations ($\hat{U}_x|0\rangle = |0\rangle$), and that the vacuum expectation value of the field vanishes ($\langle 0|\hat{\phi}(0)|0\rangle = 0$).

We can now relate the state with momentum $\mathbf{p}$ and that with zero momentum by the unitarity transformation that implements the corresponding boost, $\hat{U}_{\mathbf{p}}$:

$$|\alpha(\mathbf{p})\rangle = \hat{U}_{\mathbf{p}}|\alpha(0)\rangle. \tag{2.21}$$

Since the operator $\hat{\phi}(0)$, being a scalar, and the vacuum state are invariant under this boost

$$\langle 0|\hat{\phi}(0)|\alpha(\mathbf{p})\rangle = \langle 0|\hat{\phi}(0)|\alpha(0)\rangle, \tag{2.22}$$

we finally obtain the famous Källen-Lehmann (KL) formula

$$\begin{aligned}\langle 0|T\hat{\phi}(x)\hat{\phi}(0)|0\rangle &= \sum_{\alpha} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\alpha)} \, e^{-ip\cdot x}\big|_{p_0=E_{\mathbf{p}}(\alpha)} \, |\langle 0|\hat{\phi}(0)|\alpha(\mathbf{p})\rangle|^2 \\ &= i\sum_{\alpha} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{|\langle 0|\hat{\phi}(0)|\alpha(\mathbf{0})\rangle|^2}{p^2 - m(\alpha)^2 + i\epsilon},\end{aligned} \tag{2.23}$$

where the last equality is easy to show by performing a contour integration over p_0.

Two observations are in order

- For each state labelled by α there is a field renormalization constant

$$Z_\alpha \equiv |\langle 0|\hat{\phi}(0)|\alpha(\mathbf{0})\rangle|^2, \tag{2.24}$$

which is the same quantity that characterizes the asymptotic states in the LSZ relation of eq. (2.16).

- The states do not have to be discrete, therefore the sum over α is really an integral. It is common to write the KL relation in terms of a spectral density:

$$\langle 0|T\hat{\phi}(x)\hat{\phi}(0)|0\rangle = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2)\Delta(x; M^2), \tag{2.25}$$

with

$$\Delta(x; M^2) \equiv i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - M^2 + i\epsilon}, \tag{2.26}$$

and

$$\rho(M^2) = \sum_{\alpha \in 1particle} (2\pi) \; Z_\alpha \; \delta(M^2 - m(\alpha)^2) + \text{continuum}. \tag{2.27}$$

2.3.2 Wick rotation

The LSZ reduction formula demonstrates that correlation functions of time-ordered products of fields:

$$W_n(t_1, \mathbf{x_1}; ...., t_n, \mathbf{x}_n) = \langle 0|\hat{\phi}(t_1, \mathbf{x}_1)...\hat{\phi}(t_n, \mathbf{x}_n)|0\rangle, \quad t_1 \geq t_2..... \geq t_n, \qquad (2.28)$$

contain *all* the physical information of the theory. These objects are therefore the primary quantities to be computed on the lattice. However this is not done in Minkowski but in Euclidean space, after an analytic continuation.

It is possible to show under general conditions that these functions can be continuously extended to analytic functions in the complex domain of the variables $t_1, ...t_n$ so that

$$\text{Im } t_1 \leq \text{Im } t_2 \leq ....\text{Im } t_n. \qquad (2.29)$$

The Euclidean correlation functions or Schwinger functions are defined as:

$$S_n(x_1, ..., x_n) = W_n(-ix_1^0, \mathbf{x}_1; ... - ix_n^0, \mathbf{x}_n), \qquad (2.30)$$

where the Euclidean times are $x_i^0 = it_i$ and

$$x_1^0 \geq x_2^0.... \geq x_n^0. \qquad (2.31)$$

The computation of these functions is sufficient to solve the theory. This Euclidean approach shows all its power in the functional integral representation that we now describe.

2.4 Functional Formulation of a Scalar Field Theory

Feynman reformulated quantum mechanics via the so-called path integral (Feynman, 1948), that allows to represent the basic time evolution operator of the quantum theory as an integral over classical paths.

2.4.1 Path integral in quantum mechanics

As stated before, the quantum operator that evolves states from time t_i to t_f is

$$\hat{U}(t_f, t_i) = e^{-i\hat{H}(t_f - t_i)}, \qquad (2.32)$$

where $\hat{H}$ is the quantum Hamiltonian.

Let us consider a system of one particle with a Hamiltonian $\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{x})$. Let us divide the time interval in a large number, N, of infinitesimal intervals of width τ:

$$t_n = t_i + n\tau, \quad n = 0, ..., N, \quad \tau \equiv \frac{t_f - t_i}{N}. \qquad (2.33)$$

We can therefore write the evolution as the composition of infinitesimal evolutions

$$\hat{U}(t_f, t_i) = \hat{U}(t_f, t_{N-1})\hat{U}(t_{N-1}, t_{N-2})....\hat{U}(t_1, t_i). \qquad (2.34)$$

At each time slice t_n we can include the identity operator as the projector on a complete basis, such as the position basis

$$\hat{1} = \int d^3x_n \, |\mathbf{x}_n\rangle\langle\mathbf{x}_n|, \tag{2.35}$$

therefore

$$\begin{aligned}\hat{U}(t_f, t_i) &= \left(\prod_{n=1}^{N-1} \int d^3x_n\right) \hat{U}(t_f, t_{N-1})|\mathbf{x}_{N-1}\rangle\langle\mathbf{x}_{N-1}|....|\mathbf{x}_1\rangle\langle\mathbf{x}_1|\hat{U}(t_1, t_i) \\ &= \left(\prod_{n=1}^{N-1} \int d^3x_n\right) \hat{T}|\mathbf{x}_{N-1}\rangle \left(\prod_{n=2}^{N-1} \langle\mathbf{x}_n|\hat{T}|\mathbf{x}_{n-1}\rangle\right) \langle\mathbf{x}_1|\hat{T},\end{aligned} \tag{2.36}$$

where we have denoted the evolution operator in each interval by the *transfer operator*, $\hat{T}$:

$$\hat{U}(t_{n+1}, t_n) = e^{-i\hat{H}\tau} \equiv \hat{T}. \tag{2.37}$$

The next step is to *define* a new transfer operator $\hat{T}_F$ that coincides with $\hat{T}$ in the limit $\tau \to 0$, and which makes it easy to evaluate the matrix elements $\langle\mathbf{x}_n|\hat{T}_F|\mathbf{x}_{n-1}\rangle$. A possible definition is

$$\hat{T}_F \equiv e^{-i\frac{\tau}{2}V(\hat{x})} \; e^{-i\tau\frac{\hat{p}^2}{2m}} e^{-i\frac{\tau}{2}V(\hat{x})}, \tag{2.38}$$

which implies

$$\begin{aligned}\langle\mathbf{x_{n+1}}|\hat{T}_F|\mathbf{x_n}\rangle &= \sqrt{\frac{m}{2\pi i\tau}} \exp\left[i\tau\left(\frac{m}{2}\left(\frac{\mathbf{x}_{n+1}-\mathbf{x}_n}{\tau}\right)^2 - \frac{V(\mathbf{x}_{n+1})+V(\mathbf{x}_n)}{2}\right)\right] \\ &= \sqrt{\frac{m}{2\pi i\tau}} e^{i\tau\mathcal{L}(t_n)},\end{aligned} \tag{2.39}$$

where the function $\mathcal{L}$ is the time-discretized version of the classical Lagrangian

$$\mathcal{L}(t) \equiv \frac{1}{2}m\left(\frac{d\mathbf{x}}{dt}\right)^2 - V(\mathbf{x}), \tag{2.40}$$

and $\mathbf{x}(t_n) = \mathbf{x}_n$.

Finally, the evolution operator is given by

$$\begin{aligned}\langle\mathbf{x_f}|\hat{U}(t_f, t_i)|\mathbf{x_i}\rangle &= \lim_{N\to\infty} \left(\sqrt{\frac{m}{2\pi i\tau}}\right)^N \prod_{n=1}^{N} \int d^3x_n e^{i\tau\sum_{n=0}^{N-1}\mathcal{L}(t_n)}\Bigg|_{\mathbf{x(t_f)\equiv x_f;x(t_i)\equiv x_i}} \\ &\equiv c\int \mathcal{D}x(t)\; e^{i\int_{t_i}^{t_f} dt\mathcal{L}(t)},\end{aligned} \tag{2.41}$$

where c is a a constant. This amplitude is the *path integral* over all paths that pass by the space-time points $(t_i, \mathbf{x}_i)$ and $(t_f, \mathbf{x}_f)$.

Obviously we have not proven here the equivalence between the two representations (canonical and functional), since the two definitions of the transfer operator in eqs. (2.37) and (2.38) agree only for small τ.

The path integral representation is therefore an *alternative formulation* of quantum mechanics. Clearly the link between the world of quantum operators and that of functional integrals is the transfer operator $\hat{T}_F$, the Hamiltonian being a derived quantity:

$$\hat{H}_F \equiv \frac{i}{\tau} \ln \hat{T}_F. \tag{2.42}$$

$\hat{H}_F$ and $\hat{H}$ do not coincide, although they are expected to lead to the same physics.

As we have explained above, the quantum time evolution operator can be analytically continued to imaginary time $t \to -ix_0$, and so does the path integral representation we have just introduced. The transfer operator in Euclidean space is the positive operator:

$$\hat{T}_F^E = \exp\left(-\frac{\tau}{2}V(\hat{x})\right) \exp\left(-\tau \frac{\hat{P}^2}{2m}\right) \exp\left(-\frac{\tau}{2}V(\hat{x})\right), \tag{2.43}$$

and the relation with the Euclidean Hamilton operator is therefore

$$\hat{H}_F^E = -\frac{1}{\tau}\hat{T}_F^E. \tag{2.44}$$

From here onwards we will eliminate the F and E indices for simplicity, and denote the Euclidean transfer operator by $\hat{T}$.

An important role is played in the following by the *partition function*, which can be defined as

$$\mathcal{Z} \equiv \mathrm{Tr}\left[\hat{U}(T/2, -T/2)\right] \equiv \lim_{N\to\infty} \mathrm{Tr}\left[\hat{T}^N\right] = \int_{PBC} \mathcal{D}x(t) e^{-\int_{-T/2}^{T/2} \mathcal{L} dt}, \tag{2.45}$$

where PBC stands for periodic boundary conditions, since now the integration is over all classical paths that are periodic, i.e $\mathbf{x}_i = \mathbf{x}_f$ in eq (2.41) and we sum over $\mathbf{x}_i$.

2.4.2 Path integral in Quantum Field Theory

We have reviewed the canonical quantization of a scalar field in section 2.2, which amounts to considering not one but an infinite number of quantum operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{P}}$, one pair for each point in space:

$$\left\{\hat{x}_i, \hat{P}_i\right\} \to \left\{\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{x})\right\}, \quad \left[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] = i\delta(\mathbf{x}-\mathbf{y}), \tag{2.46}$$

satisfying the canonical commutation relations.

The quantum Hamiltonian is given by (see eq. (2.10)):

$$\hat{H} \equiv \int d\mathbf{x} \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + V(\hat{\phi}) \right]. \tag{2.47}$$

The equivalent of the complete position basis is now[2]

$$\hat{1} = \int \prod_{\mathbf{x}} d\phi(\mathbf{x}) |\phi\rangle\langle\phi|, \tag{2.48}$$

where the states $|\phi\rangle$ are the eigenstates of the field operator

$$\hat{\phi}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle. \tag{2.49}$$

Following the same steps as in the case of one degree of freedom, we can represent the time evolution operator in Euclidean time by discretizing time as before in terms of a transfer operator:

$$\hat{U}(t_f, t_i) = \hat{U}(t_f, t_{N-1})\hat{U}(t_{N-1}, t_{N-2})....\hat{U}(t_1, t_i) \equiv \hat{T}^N, N\tau = t_f - t_i. \tag{2.50}$$

$$\hat{U}(t_f, t_i) = \int \prod_{n=1}^{N-1} d\phi_n(\mathbf{x}_n)\ \hat{T}|\phi_{N-1}\rangle\langle\phi_{N-1}|\hat{T}...|\phi_1\rangle\langle\phi_1|\hat{T}. \tag{2.51}$$

The transfer operator is the analogous of eq. (2.38) and can be defined as:

$$\hat{T} = \exp\left(-\frac{\tau}{2}\hat{H}_V\right)\exp\left(-\tau\hat{H}_K\right)\exp\left(-\frac{\tau}{2}\hat{H}_V\right), \tag{2.52}$$

where

$$\hat{H}_V \equiv \int d\mathbf{x} \left[\frac{1}{2}(\nabla\hat{\phi})^2 + V(\hat{\phi}) \right], \tag{2.53}$$

$$\hat{H}_K \equiv \int d\mathbf{x} \frac{1}{2}(\hat{\pi})^2. \tag{2.54}$$

We can compute the matrix elements of this transfer operator easily (the operator $\hat{H}_V$ is diagonal in the ϕ basis, while $\hat{H}_K$ is diagonal in the momentum basis) and the result is

$$\begin{aligned}\langle\phi_{n+1}|\hat{T}|\phi_n\rangle &= \exp\left[-\frac{\tau}{2}\int d^3x \left(\frac{\phi_{n+1}(\mathbf{x}) - \phi_n(\mathbf{x})}{\tau}\right)^2 + (\nabla\phi_n)^2 + \frac{V(\phi_n) + V(\phi_{n+1})}{2} \right] \\ &= \exp\left(-\tau\mathcal{L}(\phi_n)\right),\end{aligned} \tag{2.55}$$

where $\mathcal{L}$ is the time-discretized version of the classical Euclidean Lagrangian, eq. (2.9)

$$\langle\phi_f|\hat{U}(t_f, t_i)|\phi_i\rangle = \lim_{N\to\infty} \int \left[\prod_{n=0}^{N} \prod_{\mathbf{x}_n} d\phi_n(\mathbf{x_n}) \right] \exp\left(-\tau\sum_{n=0}^{N}\mathcal{L}(\phi_n))\right)$$

[2]In the Schrödinger picture, the wave function is no longer a function of $\mathbf{x}$ but of $\phi(\mathbf{x})$.

$$\equiv \int_{\substack{\phi(\mathbf{x},t_i)=\phi_i(\mathbf{x})\\ \phi(\mathbf{x},t_n)=\phi_f(\mathbf{x})}} \mathcal{D}\phi \exp\left(-\int dt\mathcal{L}(\phi)\right). \tag{2.56}$$

We can also define the partition function as

$$\mathcal{Z} = \lim_{N\to\infty} \text{Tr}\left[\hat{T}^N\right] = \int_{PBC} \mathcal{D}\phi\ e^{-\int dt\mathcal{L}(\phi)} = \int_{PBC} \mathcal{D}\phi\ e^{-\mathcal{S}[\phi]}, \tag{2.57}$$

where

$$S[\phi] = \int dt\mathcal{L}(\phi) = \int d^4x \left\{\frac{1}{2}\left(\partial_\mu\phi(x)\right)^2 + V(\phi(x))\right\}, \tag{2.58}$$

is the classical Euclidean action.

2.4.3 Correlation functions in the functional formalism

We are interested in correlation functions. We can easily derive their functional representation similarly by noting that for any operator

$$\langle 0|\hat{O}(\mathbf{x},t)|0\rangle = \lim_{T\to\infty} \frac{\text{Tr}\left[\hat{O}e^{-\hat{H}T}\right]}{\text{Tr}\left[e^{-\hat{H}T}\right]} = \lim_{T\to\infty} \frac{\text{Tr}\left[\hat{O}e^{-\hat{H}T}\right]}{\mathcal{Z}}, \tag{2.59}$$

provided $|0\rangle$ is the lowest energy state, since the contribution to the trace of the excited states is exponentially suppressed.

Then we can write the time-ordered correlation function

$$S_n = \langle 0|\hat{\phi}(\mathbf{x}_1,t_1)....\hat{\phi}(\mathbf{x}_n,t_n)|0\rangle = \lim_{T\to\infty} \text{Tr}\left[\hat{\phi}(\mathbf{x}_1,t_1)...\hat{\phi}(\mathbf{x}_n,t_n)e^{-\hat{H}T}\right]/\mathcal{Z}. \tag{2.60}$$

and applying the same procedure of discretizing time we find the functional representation of the n-point function

$$S_n = \frac{\int_{PBC}\mathcal{D}\phi\ e^{-S[\phi]}\phi(\mathbf{x}_1,t_1)....\phi(\mathbf{x}_n,t_n)}{\int_{PBC}\mathcal{D}\phi\ e^{-S[\phi]}} \equiv \langle\phi(x_1)....\phi(x_n)\rangle, \tag{2.61}$$

where the integrals are over periodic classical fields, as defined above.

Some useful definitions in the context of perturbation theory are:

- the generating functional of correlation functions is

$$Z[J] = \langle e^{\int d^4x J(x)\phi(x)}\rangle, \tag{2.62}$$

where we have introduced an external source density $J(x)$ so that

$$\frac{\delta}{\delta J(x_1)}....\frac{\delta}{\delta J(x_n)}Z[J]\bigg|_{J=0} = \langle\phi(x_1)...\phi(x_n)\rangle. \tag{2.63}$$

We define a functional derivative as

$$\frac{\delta}{\delta J(x)} J(y) = \delta(x-y), \quad \frac{\delta}{\delta J(x)} \int d^4 y J(y)\phi(y) = \phi(x). \tag{2.64}$$

It is easy to compute $Z[J]$ in the scalar field theory we are considering for the free case, ie. for $V(\phi) = \frac{m_0^2}{2}\phi^2$. The path integral is Gaussian and the result is

$$Z[J] = \exp\left(\frac{1}{2}\int d^4x d^4y J(x) K^{-1}(x,y) J(y)\right), \tag{2.65}$$

where

$$K \equiv -\partial^\mu \partial_\mu + m_0^2, \tag{2.66}$$

is a linear operator acting in the space of real scalar fields. The free propagator is

$$\langle \phi(x)\phi(y)\rangle = \left.\frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)}\right|_{J=0} = K^{-1}(x,y) = \int d^4p \, \frac{e^{i(x-y)}}{p^2 + m_0^2}. \tag{2.67}$$

- The generating functional of connected correlation functions $W[J] \equiv \ln Z[J]$ satisfies:

$$\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W[J] = \langle \phi(x_1) \ldots \phi(x_n)\rangle_{conn}. \tag{2.68}$$

- The generating functional of vertex functions, which are connected and one-particle amputated correlation functions, also called one-particle irreducible or 1PI [3], can be obtained from the Legendre transform of $W[J]$:

$$\Gamma[\Phi] = W[J] - \left.\int d^4x J(x)\Phi(x)\right|_{J[\Phi]} \tag{2.69}$$

where $J[\Phi]$ is defined from the solution of equation

$$\frac{\delta W[J]}{\delta J(x)} = \Phi(x). \tag{2.70}$$

The functional derivatives of $\Gamma[\Phi]$ generate the 1PI correlation functions

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \frac{\delta}{\delta \Phi(x_1)} \cdots \frac{\delta}{\delta \Phi(x_n)} \Gamma[\Phi] = \langle \phi(x_1)\phi(x_2) \ldots \phi(x_n)\rangle_{conn,1PI} \tag{2.71}$$

or vertex functions that represent the interaction vertices in the Lagrangian and are therefore the basic objects in the renormalization procedure. More details can be found in standard books (Peskin and Schroeder, 1995).

All these generating functionals are easy to find in the free case, but not in the interacting case. At this point one can follow two approaches:

[3] These are the correlation functions which cannot be made disconnected by cutting out one particle propagator.

- Perturbation theory
- A non-perturbative evaluation of the correlation functions, which can be achieved via a discretization of space-time, known as the lattice formulation. This Euclidean functional formulation of QFT provides a link between QFT and a statistical system. After the discretization of space-time, the functional integrals of eqs. (2.57) become finite-dimensional ones, and in many cases can be treated by statistical importance sampling methods. I refer to the lectures of M. Lüscher (Lüscher, 2009) for a general discussion of these methods.

2.5 Symmetries and Ward Identities

Noether's theorem establishes the connection between continuous symmetries of the Lagrangian and conserved currents. In the functional formulation, symmetries of the Lagrangian imply relations between correlation functions that are usually referred to as *Ward-Takahashi identities* (Ward, 1950; Takahashi, 1957). These identities are easy to derive at tree level and can be shown to hold also at the quantum level (Peskin and Schroeder, 1995).

Let us consider an infinitesimal local field transformation of the form:

$$\phi(x) \to \phi(x) + \epsilon_a(x)\delta_a\phi(x), \tag{2.72}$$

which will usually correspond to a unitary transformation. The Lagrangian changes at first order by

$$\delta\mathcal{L}[\phi] = \frac{\delta\mathcal{L}}{\delta\epsilon_a(x)}\epsilon_a(x) + \frac{\delta\mathcal{L}}{\delta\partial_\mu\epsilon_a(x)}\partial_\mu\epsilon_a(x) + \mathcal{O}(\epsilon^2). \tag{2.73}$$

Now let's consider the generating functional

$$Z[J] = \int D\phi e^{-S[\phi]+\int d^4x J(x)\phi(x)}, \tag{2.74}$$

on which we can perform the change of variables of eq. (2.72)

$$Z[J] = \int D\phi' e^{-S[\phi']+\int d^4x J(x)\phi'(x)} = Z[J] + \delta Z[J]. \tag{2.75}$$

Since this should be true for arbitrary $\epsilon_a(x)$, and assuming the measure does not change ($D\phi' = D\phi$) we have

$$\frac{\delta Z[J]}{\delta\epsilon^a(x)} = \int D\phi e^{-S[\phi]+\int d^4x J(x)\phi(x)} \left(\partial_\mu \mathcal{J}^a_\mu - \left.\frac{\delta\mathcal{L}}{\delta\epsilon_a(x)}\right|_{\epsilon=0} + J(x)\delta_a\phi(x) \right) = 0, \tag{2.76}$$

where $\mathcal{J}^a_\mu$ coincides with the classically conserved Noether current,

$$\mathcal{J}^a_\mu(x) \equiv \left.\frac{\delta\mathcal{L}(\phi + \epsilon^a\delta_a\phi)}{\delta\partial_\mu\epsilon_a(x)}\right|_{\epsilon=0}. \tag{2.77}$$

The n-th functional derivatives with respect to the external sources, J, of the functional in eq. (2.76) give relations between the correlation functions of the following type:

$$\frac{\partial}{\partial x_\mu}\langle\phi(x_1)\phi(x_2)...\phi(x_n)\mathcal{J}^a_\mu(x)\rangle = \langle\phi(x_1)\phi(x_2)...\phi(x_n)\left.\frac{\delta\mathcal{L}}{\delta\epsilon_a(x)}\right|_{\epsilon=0}\rangle$$
$$-\sum_i \delta(x_i - x)\langle\phi(x_1)..\delta_a\phi(x_i)..\phi(x_n)\rangle, \qquad (2.78)$$

where the second term on the right are contact terms which vanish if $x \neq x_1, ..., x_n$. For more details on the derivation of these identities see for example (Collins, 1984).

2.6 Perturbation Theory in the Functional Formalism

Correlation functions in the interacting case, i.e. for

$$V(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{\lambda}{4!}\phi^4 \qquad (2.79)$$

can be obtained by perturbing in λ. We just need to separate the free and interacting parts of the classical action:

$$S[\phi] = S^{(0)}[\phi] + S^{(1)}[\phi], \qquad (2.80)$$

with

$$S^{(0)}[\phi] \equiv \int d^4x \left\{\frac{1}{2}\left[(\partial_\mu\phi(x))^2 + m_0^2\phi^2\right]\right\}, S^{(1)}[\phi] = \int d^4x \frac{\lambda}{4!}\phi^4. \qquad (2.81)$$

The generating functional can therefore be Taylor-expanded in the coupling constant, λ:

$$Z[J] = \frac{\langle e^{\int d^4x J(x)\phi(x)} e^{-S^{(1)}[\phi]}\rangle_0}{\langle e^{-S^{(1)}[\phi]}\rangle_0}$$
$$\equiv \frac{\int \mathcal{D}\phi e^{-S^{(0)}[\phi]+\int d^4x J(x)\phi(x)} \sum_n \frac{1}{n!}\left(-S^{(1)}[\phi]\right)^n}{\int \mathcal{D}\phi e^{-S^{(0)}[\phi]} \sum_n \frac{1}{n!}\left(-S^{(1)}[\phi]\right)^n}, \qquad (2.82)$$

where $\langle\rangle_0$ is the average with respect to the unperturbed theory and therefore can be evaluated in terms of the free generating functional of eq. (2.65). The n-th Schwinger function is given by

$$S_n = \frac{\langle\phi(x_1)\phi(x_2)...\phi(x_n)e^{-S^{(1)}[\phi]}\rangle_0}{\langle e^{-S^{(1)}[\phi]}\rangle_0}, \qquad (2.83)$$

and a similar Taylor expansion in λ allows to compute S_n in terms of free correlation functions. Three observations are in order:

- Wick's theorem holds. All contributions can be obtained from functional derivatives of the free path integral, $Z^{(0)}[J]$, evaluated at $J = 0$. Therefore Wick's theorem is reproduced because $Z^{(0)}[J]$ is quadratic in the currents and therefore the fields have to be paired up in propagators to give a non-vanishing contribution:

$$\langle \phi(x_1)\phi(x_2)....\phi(x_{2n})\rangle_0 = \sum_{\text{perm}} \langle \phi(x_1)\phi(x_2)\rangle_0....\langle \phi(x_{2n-1})\phi(x_{2n})\rangle_0. \quad (2.84)$$

  Correlation functions are therefore obtained from products of propagators.
- The denominator in eq. (2.83) ensures that all contributions with disconnected parts that do not contain any external leg cancel (i.e. vacuum polarization diagrams).
- For each insertion of $S^{(1)}[\phi]$ there is an integration over space-time that can give rise to ultraviolet divergences (UV).

A similar perturbative expansion can be trivially defined for the generating functionals of connected and 1PI diagrams.

2.7 Perturbative renormalizability

In order to ensure the UV finiteness of the perturbative contributions, it is sufficient to consider the 1PI diagrams, where the propagators attached to the external legs are amputated. Let us consider a general diagram of an N-th 1PI correlation function in momentum space for the scalar theory, eq. (2.79). The contribution of a diagram with I internal lines (i.e. propagators linking two vertices) and L loops is generically of the form:

$$\Gamma^{(N)}(p_1, ..., p_N) \sim \int \prod_{l=1}^{L} d^4 q_l \prod_{i=1}^{I} \frac{1}{k_i(q_l, p_j)^2 + m^2}, \quad (2.85)$$

where the q_l stand for the L loop momenta, p_j for the N external momenta and k_i are the momenta of the I internal lines, that can in general be writen as linear combinations of the external and loop momenta. The loop momentum integrals give rise to UV divergences. If these integrals are cutoff at some scale Λ, the diagram behaves as $\sim \Lambda^\omega$ when Λ is scaled to ∞, where ω is the power of the leading divergence, also called the *superficial degree of divergence*. Scaling the loop momenta with Λ in eq. (2.85), the following relation follows:

$$\omega \equiv 4L - 2I. \quad (2.86)$$

ω must therefore be negative for the diagram to be finite, although this condition is not sufficient to ensure finiteness.

There is a topological relation between I, the number of vertices V and external legs N of the diagram:

$$2I + N = 4V, \quad (2.87)$$

since each vertex involves four fields and each leg is either external or linked to another internal line.

Finally the number of loops, L, is related to V and N. Each propagator involves an integral over momentum, eq. (2.67). Each vertex involves an integration over space-time, giving rise to $\delta(\sum_i p_i)$, where the sum is over all momenta attached to the vertex. One of these deltas corresponds to the conservation of the external momenta, while the others allow to reduce $V-1$ of the loop integrations. Therefore the number of loops of the diagram satisfies, using eq. (2.87),

$$L = I - V + 1 = V - N/2 + 1, \tag{2.88}$$

and substituting eqs. (2.87) and (2.88) in eq. (2.86) we find

$$\omega = 4 - N. \tag{2.89}$$

ω does not depend on the number of loops or vertices. It is fixed by the number of external legs. Only 1PI diagrams with $N = 2, 4$ might have a non-negative degree of divergence. It can be shown that the UV divergences in these diagrams give contributions to the vertex functions of the form

$$\delta\Gamma^{(2)}[\Phi] = A\partial_\mu\Phi\partial_\mu\Phi + B\Phi^2 \tag{2.90}$$

$$\delta\Gamma^{(4)}[\Phi] = C\Phi^4, \tag{2.91}$$

where A, B, C are divergent, but since they have the same structure as the terms already present in the Lagrangian, they can be reabsorbed in a redefinition of m_0^2, λ and the normalization of the field itself. For this reason, we say that this theory is *perturbatively renormalizable*.

More generically, we can consider a theory where $S^{(1)}$ has other interactions such as

$$S^{(1)}[\phi] = \frac{\lambda}{4!}\phi^4 + \frac{\lambda'}{6!}\phi^6 + ... \tag{2.92}$$

while λ has no mass dimension, the additional couplings in general do, e.g. $[\lambda'] = -2$.

Let us consider more generally a vertex with N_∂ derivatives and N_ϕ fields. The corresponding coupling, g_V, must have mass dimension

$$[g_V] = 4 - N_\phi - N_\partial. \tag{2.93}$$

We can repeat the power-counting exercise above to evaluate the superficial degree of divergence of a vertex function that contains V vertices of this type and we find that the relations eqs. (2.87), (2.88) are modified to

$$\omega = 4L - 2I + N_\partial V \quad 2I + N = N_\phi V \quad L = I - V + 1, \tag{2.94}$$

and therefore

$$\omega = 4 - N - [g_V]V. \tag{2.95}$$

We find a very different behaviour as the order of the perturbative expansion grows depending on the sign of $[g_V]$:

$[g_V] > 0$	diagrams become less divergent with V: *superrenormalizable theory*
$[g_V] = 0$	the divergence does not depend on V: *renormalizable theory*
$[g_V] < 0$	divergences for larger N as V grows: *non-renormalizable theory*

Even if one considers only renormalizable theories, the proof of perturbative renormalizability is rather involved, because a diagram with $\omega < 0$ does not have to be finite. In general there are subdivergences (that is divergences that show up when a subset of all the internal momenta are scaled with Λ). The proof of renormalizability in the continuum takes therefore the following steps:

- Prove a power counting theorem to characterize divergent and finite diagrams
- Recursive procedure to subtract subdivergences: e.g. in the BPHZ (Bogoliubov and Parasiuk, 1957; Hepp, 1966; Zimmermann, 1969) subtraction scheme, the superficial degree of divergence of a diagram is reduced by subtracting the Taylor expansion of the diagram in the external momenta up to order equal the degree of the polynomial. A forest formula establishes the recursive procedure to subtract subdivergences.
- All-orders proof

The conclusions to all orders in perturbation theory are the same as those based on the superficial degree of divergence. For more details about perturbative renormalizability we refer to P. Weisz's lectures (Weisz, 2009).

2.8 Wilsonian renormalization group

The old concept of renormalizability which looked like a sacred requirement of any sensible quantum field theory is now updated. Thanks to Wilson and others we know now that there is nothing special about a bare Lagrangian that is renormalizable. In fact the consequence of the point of view of assuming the existence of a fundamental cutoff (such as the one existing in a theory defined on the lattice) is that renormalizability is an emergent effective phenomenon. If such a theory describes correlations that tend to infinity in units of the cutoff, it can be accurately represented by a renormalizable theory, as long as we are interested in describing physics at scales of the order of this long correlation length. For a classical reference see (Wilson and Kogut, 1974) and references therein.

2.8.1 Renormalization group transformations

K. Wilson studied the connection of renormalizability and critical phenomena via his celebrated *renormalization group transformations*. Let us assume that we have a real cutoff, such as a space-time lattice spacing $a = \Lambda^{-1}$, as we will see later. Taking the continuum limit $a \to 0$ is therefore like taking the cutoff to infinity, and the hope is that a finite limit exists, in which physical scales stay finite and therefore

$$m_{phys} a \to 0. \tag{2.96}$$

Seen as a statistical system this implies that the correlation length (rate of the exponential decay of the two-point correlator), $\xi \sim m_{phys}^{-1}$, goes to infinity in units of the lattice spacing

$$\xi/a \to \infty, \tag{2.97}$$

and this is what we call in statistical mechanics a *critical point*. The continuum limit of a QFT must therefore be a critical point.

It is an empirical fact that many systems near critical points behave in similar ways, this is what is called *universality* (the long range properties of many systems do not depend on the details of the microscopic interactions). It was the contribution of Wilson and others that established the link

Universality in critical statistical systems $\leftrightarrow$ Renormalizability in QFT

Both phenomena can be understood in terms of *fixed-points* of the renormalization group.

Let us suppose that we have a lattice scalar theory on a lattice of spacing a which describes physics scales $m \ll a^{-1}$. The most general theory that is local can be written as

$$S(a) = \sum_\alpha g_\alpha(a) \sum_x O_\alpha(\phi(x)), \tag{2.98}$$

where O_α are local operators (of the field and its derivatives) with arbitrary dimension that respect the lattice symmetries. This is a very complicated system with many coupled degrees of freedom, however if we are interested only in the long-distance properties, many of the degrees of freedom (those at short distance or large momenta) induce effects that can be absorbed in a change in the couplings g_α, as we will see.

In order to understand what happens when we take the limit $a \to 0$ keeping the physical scale fixed, we can follow Wilson's recipe and do it in little steps. We consider a series of lattice spacings that decrease by a factor $1 - \epsilon$ at a time:

$$a \geq a_1 \geq a_2 ... \geq a_n = (1-\epsilon)^n a, \quad \epsilon \ll 1. \tag{2.99}$$

We want to compare the actions defined in the series of lattices and we do this by defining, at each step n, an effective action at the original scale a at each step $S^{(n)}(a)$. This action is obtained from the n-th action at the scale a_n, after integrating out recursively the extra degrees of freedom that appear at each step. These are short-ranged (momentum scales between a_{n-1}^{-1} and a_n^{-1}), and therefore result in a local action, which must then have the same generic form of eq. (2.98), but with different couplings in general:

$$S^{(n)}(a) = \sum_\alpha g_\alpha^{(n)}(a) \sum_x O_\alpha(\phi(x)). \tag{2.100}$$

We call a *renormalization group (RG) transformation*, the function that defines the change in the couplings:

$$R_\alpha : g_\alpha^{(n)} \to g_\alpha^{(n+1)} \quad g_\alpha^{(n+1)} = R_\alpha(g^{(n)}). \tag{2.101}$$

Obviously we can make this transformation a continuous one and then we talk about the RG flow of the coupling constants. While the couplings change we are changing the physics obviously, but if we perform sufficiently many transformations we can hit a fixed-point if it exists. A *fixed-point* corresponds to some point in coupling space g^*_α such that

$$R_\alpha(g^*) = g^*_\alpha. \tag{2.102}$$

It is at these points that physics would no longer change as we move towards the continuum limit, since the action remains unchanged. The fixed-points are therefore critical points:

$$\lim_{n\to\infty} m_{phys}(g^*)a_n \to 0, \tag{2.103}$$

unless the physical critical mass diverges, which would be uninteresting for a QFT.

Now it turns out that such fixed-points, if they exist, are rather universal, because they can be approached by tuning just a few parameters , called relevant couplings. A priori, one could imagine having to tune all the couplings $\alpha = 1, .., \infty$ to reach a given fixed-point, but this is usually not the case and this is the essence of renormalizability and universality. Near a fixed-point the evolution of the couplings reads at linear order

$$g_\alpha^{(n+1)} - g^*_\alpha = \sum_\beta \left.\frac{\partial R_\alpha}{\partial g_\beta}\right|_{g^*} (g_\beta^{(n)} - g^*_\beta), \tag{2.104}$$

so the distance to the fixed-point $\Delta g^{(n)}$ changes according to the following equation:

$$\Delta g_\alpha^{(n+1)} = \sum_\beta M_{\alpha\beta}\Delta g_\beta^{(n)}, \qquad M_{\alpha\beta} \equiv \left.\frac{\partial R_\alpha}{\partial g_\beta}\right|_{g^*}. \tag{2.105}$$

We can find different situations depending on the eigenvalues, λ, of the matrix M:

$\lambda > 1$	$\Delta g_\alpha^{(n)}$ increases as $n \to \infty$	α is a relevant direction
$\lambda = 1$	$\Delta g_\alpha^{(n)}$ stays the same as $n \to \infty$	α is a marginal direction
$\lambda < 1$	$\Delta g_\alpha^{(n)}$ decreases as $n \to \infty$	α is an irrelevant direction

In the first case the distance to the fixed point grows in the corresponding direction, these are *relevant couplings* that would need to be tuned. In the third case, the distance to the fixed-point decreases and these are *irrelevant couplings*. In the second case, the couplings are called *marginal* and might need tuning or not depending on subtle quantum effects that always make λ slightly different from one. The fact that the number of relevant directions is finite and usually small is behind the two related properties: universality of the fixed-point and the renormalizability of QFT.

Gaussian Fixed Point. We will make this discussion a bit more explicit by considering the Gaussian fixed-point of scalar theories. First we note that the free massless point of a scalar theory is a fixed-point. Consider the action

$$S(a) = \int_{BZ(a)} \frac{d^4p}{(2\pi)^4}\frac{1}{2}\phi(-p)p^2\phi(p), \tag{2.106}$$

where $BZ(a)$ is the Brillouin zone $[-\pi/a, \pi/a]$ in each momentum direction.

When we do the first RG transformation we start with the same action but in a lattice of spacing $a_1 = (1-\epsilon)a$. Since the fields at different momenta are independent variables, we can integrate over those at momenta $\pi/a \leq |p_\mu| \leq \pi/a_1$ so that the partition function:

$$\mathcal{Z}^{(1)} = \int \prod_{p\in BZ(a_1)} d\phi(p) e^{-\int_{BZ(a_1)} \frac{d^4p}{(2\pi)^4}\frac{1}{2}\phi(-p)p^2\phi(p)} \tag{2.107}$$

$$= C\int \prod_{p\in BZ(a)} d\phi(p) e^{-\int_{BZ(a)} \frac{d^4p}{(2\pi)^4}\frac{1}{2}\phi(-p)p^2\phi(p)}. \tag{2.108}$$

where C is some constant that comes from the integration of the momentum modes of $BZ(a_1)$ that lay out of $BZ(a)$. The effective action after integrating the high frequency modes up to scale a^{-1} is therefore $S^{(1)}(a) = S(a)$. The original action is a fixed-point of the renormalization group. Note that since there is no mass term, it is also a critical point, as expected.

Now we can see why the gaussian fixed-point is the one responsible for the renormalizability of $\lambda\phi^4$. We start with an arbitrary lattice action that is quadratic in the fields, but including all terms that have the lattice symmetries.

$$S(a) = \int_{BZ(a)} \frac{d^4p}{(2\pi)^4}\frac{1}{2}\phi(-p)\left(p^2 + \frac{1}{a^2}m_0^2 + g_1 a^2 p^4 + ...\right)\phi(p) \tag{2.109}$$

where we have expressed all the couplings in units of the lattice spacing to make them dimensionless:

$$[m_0] = [\alpha] = ... = 0. \tag{2.110}$$

This action is also diagonal in momentum space and therefore the integration over the momentum modes in a slice of momenta in $BZ(a_1)$ and out of $BZ(a)$ can be done as before so the action for the modes up to a^{-1} is the same, but erasing the high-momentum modes, ie:

$$S^{(1)}(a) = \int_{BZ(a)} \frac{d^4p}{(2\pi)^4}\frac{1}{2}\phi(-p)\left(p^2 + \frac{1}{a^2}\left(\frac{a}{a_1}\right)^2 m_0^2 + g_1 a^2\left(\frac{a_1}{a}\right)^2 p^4 + ...\right)\phi(p), \tag{2.111}$$

therefore the action is no longer a fixed-point, because all the couplings except the kinetic term have changed:

$$\begin{pmatrix} {m_0^{(1)}}^2 \\ g_1^{(1)} \\ ... \end{pmatrix} = M\begin{pmatrix} m_0^2 \\ g_1 \\ ... \end{pmatrix}, \quad M = diag\left((1-\epsilon)^{-2}, (1-\epsilon)^2, ...\right), \tag{2.112}$$

The only eigenvalue of M which is above one is the first one, therefore there is one relevant direction, that of m_0^2 and all the rest are irrelevant. After a large number of

RG transformations (as we approach the continuum limit) these directions disappear. On the other hand, m_0^2 which fixes the physical mass gap grows, therefore it needs to be tuned to remain finite in the continuum limit. Therefore the continuum limit of this theory, even if it has non-renormalizable terms should correspond to a free massive renormalizable scalar QFT.

Finally in the fully interacting case, the situation is more complicated, but still near the gaussian fixed-point (sufficiently small couplings) the continuum limit corresponds to a renormalizable scalar field theory. In this case the action contains all terms, including interactions

$$S(a) = \sum_x \partial_\mu\phi\partial_\mu\phi + \frac{1}{2a^2}m_0^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\lambda'}{6!}\phi^6 + g_1 a^2\phi\partial^4\phi + ... \tag{2.113}$$

Now the integration over the momentum shell $\pi/a \leq |p_\mu| \leq \pi/a_1$ cannot be done analytically. But for sufficiently small couplings it can be done in perturbation theory, see for example (Peskin and Schroeder, 1995). It gives

$$S^{(1)}(a) = \sum_x Z^{(1)}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2a^2}{m_0^{(1)}}^2\phi^2 + \frac{\lambda^{(1)}}{4!}\phi^4 + a^2\frac{\lambda'^{(1)}}{6!}\phi^6 + g_1^{(1)}a^2\phi\partial^4\phi + ..., \tag{2.114}$$

where

$$Z^{(1)} = 1 + \mathcal{O}(\lambda^2), \tag{2.115}$$

and

$${m_0^{(1)}}^2 = (m_0^2 + \delta m_0^2)(1-\epsilon)^{-2}, \tag{2.116}$$

$$\lambda^{(1)} = \lambda + \delta\lambda, \tag{2.117}$$

$$\lambda'^{(1)} = (\lambda' + \delta\lambda')(1-\epsilon)^2, \tag{2.118}$$

$$g_1^{(1)} = (g_1 + \delta g_1)(1-\epsilon)^2. \tag{2.119}$$

All δ terms depend on the couplings $\lambda, \lambda', ...$, but vanish for small enough couplings. Therefore for small enough couplings, the matrix M in this case has one relevant direction, many irrelevant ones and just one marginal. It is for this marginal direction that the value of $\delta\lambda$, even if small, is important since it determines the fate of this direction. At lowest order of perturbation theory it is

$$\delta\lambda = \frac{3\lambda^2}{16\pi^2}\ln(1-\epsilon) < 0, \tag{2.120}$$

therefore $\lambda^{(1)} < \lambda$ and the direction is marginally irrelevant. The change is much slower than for an irrelevant direction since it is only logarithmic. The continuum theory is therefore again a massive free scalar theory, at least within this perturbative analysis.

Summarizing Wilson's approach to renormalization shows the following intuitive physical picture:

QFT with a cutoff $\leftrightarrow$ Statistical system near criticality
Renormalized QFT $\leftrightarrow$ Statistical system at a fixed-point

This picture is of course an essential ingredient for the definition of QFT on a lattice, because it implies that we do not have to worry about the precise definition of $S(a)$, the continuum limit will approach the fixed-point of the statistical system nevertheless. We need to ensure however that the fixed-point corresponds to the QFT we want to describe. For this we need to make sure that

- the action has the right degrees of freedom
- it is local
- has the right symmetries to flow to the desired fixed-point (for example if we break some symmetry we might artificially increase the number of relevant directions)

Under these very general assumptions we are otherwise free to make our choice.

Exercise 1.1 Consider the 1D Ising model with an action:

$$S = -\beta \sum_x \sigma_x \sigma_{x+1} \quad \beta > 0. \tag{2.121}$$

where the spin variables $\sigma_x = \pm 1$. Identify the quantum operator and the transfer operator for this model. Diagonalize the transfer operator. Compute the correlator from this result, ie.

$$\langle \sigma_x \sigma_y \rangle = \lim_{N \to \infty} \mathrm{Tr}[\hat{T}^{N-(x-y)} \hat{\sigma} \hat{T}^{(x-y)} \hat{\sigma}] / \mathrm{Tr}[\hat{T}^N], \tag{2.122}$$

show that the correlation length is

$$\xi^{-1} = -\ln \tanh \beta, \tag{2.123}$$

and therefore only diverges at $\beta = \infty$ (zero temperature).

3

Lattice Scalar Field Theory

The definition of a scalar quantum field theory on the lattice assumes that the field lives in a discretized space-time. The simplest choice is to consider the lattice spacing a to be the same in all space-time directions, that is a cubic lattice:

$$\phi(x) \qquad x = na \quad n = (n_0, n_1, n_2, n_3) \quad n_i \in Z^4. \tag{3.1}$$

Therefore

$$\int dx_i \to a \sum_{n_i \in Z} \qquad \int d^4x \to a^4 \sum_x \equiv a^4 \sum_{n \in Z^4} . \tag{3.2}$$

Canonical quantization goes through identically, the only change is that the labeling of degrees of freedom is discrete and not continuous.

The Fourier transform therefore becomes a Fourier series. Any function defined on a cubic lattice, $F(na)$, has a Fourier transform which is periodic in the Brillouin zone (BZ):

$$\tilde{F}(p) = a^4 \sum_n e^{-ipna} F(na) \quad \tilde{F}(p) = \tilde{F}\left(p + \frac{2\pi}{a} m\right), \ \ m \in Z^4. \tag{3.3}$$

It is easy to invert the relation of eq. (3.3):

$$\int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} e^{ipna} \tilde{F}(p) = F(na). \tag{3.4}$$

Therefore lattice four-momenta are cutoff at scale $|p_i| \leq \pi/a$ and therefore the inverse lattice spacing, a^{-1}, is also an energy cutoff, i.e. the theory is regularized.

A very useful formula is Poisson's summation formula:

$$\sum_{n \in Z^4} e^{inz} = (2\pi)^4 \sum_{n \in Z^4} \delta(z - 2\pi n) \equiv (2\pi)^4 \delta_P(z). \tag{3.5}$$

The functional approach to quantization in Euclidean space-time involves the partition function.

$$\mathcal{Z} = \int \mathcal{D}\phi \ e^{-S[\phi]}, \quad \mathcal{D}\phi \to \prod_x d\phi(x), \tag{3.6}$$

and $S[\phi]$ is some discretized version of the action of eq. (2.58), which is not unique. According to Wilson's RG all actions should be equivalent in the continuum limit provided they satisfy the same symmetries (in this case $\phi \leftrightarrow -\phi$). The simplest choice is:

$$S[\phi] \to a^4 \sum_x \left\{ \frac{1}{2}\hat{\partial}_\mu \phi(x) \hat{\partial}_\mu \phi(x) + \frac{1}{2} m_0^2 \phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\}, \tag{3.7}$$

where we have defined the *forward lattice derivative*

$$\hat{\partial}_\mu \phi(x) \equiv \frac{1}{a}\left(\phi(x+\hat{\mu}a) - \phi(x)\right). \tag{3.8}$$

We can also define a *backward derivative*

$$\hat{\partial}^*_\mu \phi(x) \equiv \frac{1}{a}\left(\phi(x) - \phi(x-\hat{\mu}a)\right). \tag{3.9}$$

As in the continuum we can obtain the correlation functions from the generating functional

$$Z[J] \equiv \int \prod_x d\phi(x) e^{-S[\phi] + a^4 \sum_x J(x)\phi(x)} / \mathcal{Z}. \tag{3.10}$$

3.1 Free lattice scalar theory

As in the continuum, it is easy to solve the lattice theory in the free case, that is for $\lambda = 0$. We can rewrite the action as

$$S^{(0)}[\phi] = a^4 \sum_x \left\{ \frac{1}{2}\hat{\partial}_\mu \phi \hat{\partial}_\mu \phi + \frac{m_0^2}{2}\phi^2 \right\} = \frac{a^4}{2} \sum_{x,y} \phi(x) K_{xy} \phi(y), \tag{3.11}$$

with

$$K_{xy} \equiv -\frac{1}{a^2} \sum_{\hat{\mu}=0}^{3} \left(\delta_{x+a\hat{\mu} y} + \delta_{x-a\hat{\mu} y} - 2\delta_{xy}\right) + m_0^2 \delta_{xy}. \tag{3.12}$$

The corresponding generating functional is

$$Z^{(0)}[J] = e^{\frac{a^4}{2}\sum_{x,y} J_x (K^{-1})_{xy} J_y} \det\left(a^4 K\right)^{-1}, \tag{3.13}$$

where we have used $a^4 \sum_y K_{xy} K^{-1}_{yz} = \delta_{xz}$.

We can then compute the propagator:

$$\langle \phi(x)\phi(y)\rangle_0 = \frac{1}{a^8} \frac{\partial Z^{(0)}[J]}{\partial J_x \partial J_y}\bigg|_{J=0} = \frac{1}{a^4} K^{-1}_{xy}. \tag{3.14}$$

To get a more familiar expression we go to Fourier space. Using Poisson's formula eq. (3.5), after some easy manipulations we find

$$\tilde{K}_{pq} = a^8 \sum_{xy} e^{-ipx} e^{-iqy} K_{xy} = a^4 (2\pi)^4 \delta_P(p+q) \left\{ m_0^2 + \frac{2}{a^2}\sum_\mu (1 - \cos p_\mu a) \right\}$$

$$= a^4(2\pi)^4\delta_P(p+q)\left\{m_0^2 + \sum_\mu \hat{p}_\mu^2\right\}, \tag{3.15}$$

where

$$\hat{p}_\mu \equiv \frac{2}{a}\ \sin\left(\frac{p_\mu a}{2}\right) \qquad \hat{p}^2 \equiv \sum_\mu \hat{p}_\mu^2. \tag{3.16}$$

Therefore

$$K_{xy} = a^4 \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\left(\hat{p}^2 + m_0^2\right). \tag{3.17}$$

It is easy to see that the inverse is

$$\langle\phi(x)\phi(y)\rangle = a^{-4}K_{xy}^{-1} = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{\hat{p}^2 + m_0^2}. \tag{3.18}$$

Since in the free theory all correlation functions are products of propagators, this is enough to construct all correlation functions.

It is instructive to understand in this very simple context two important questions:

- what is the particle interpretation ?
- what happens in the continuum limit ?

According to the discussion in section 2.3.1, the spectral representation of the propagator at large times provides a direct link between the Euclidean formulation and the particle interpretation. Indeed we can identify the one-particle asymptotic states from the Källen-Lehmann spectral representation of the propagator, eq.(2.23):

$$\lim_{x_0\to+\infty}\langle\phi(x)\phi(0)\rangle = \sum_\alpha \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\alpha)}|\langle 0|\hat{\phi}(0)|\alpha(0)\rangle|^2 e^{-E_{\mathbf{p}}(\alpha)x_0}e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{3.19}$$

with $E_{\mathbf{p}}(\alpha) = \sqrt{m_\alpha^2 + \mathbf{p}^2}$.

Starting with the free propagator, eq. (3.18), we can perform the integral over $p_0 \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$ (contour A) using the residuum theorem (see Fig. 3.1): We consider the closed contour including the interval A , the contour B $\left[\frac{\pi}{a}, \frac{\pi}{a} + i\infty\right]$, the contour C $\left[\frac{\pi}{a} + i\infty, -\frac{\pi}{a} + i\infty\right]$ and the contour D $\left[-\frac{\pi}{a} + i\infty, -\frac{\pi}{a}\right]$. We have then

$$\int_A (...) + \int_B (...) + \int_C (...) + \int_D (...) = 2\pi i \sum_{poles} \text{Residues.} \tag{3.20}$$

By periodicity of the function in the BZ, we have

$$\int_B (...) + \int_D (...) = 0, \tag{3.21}$$

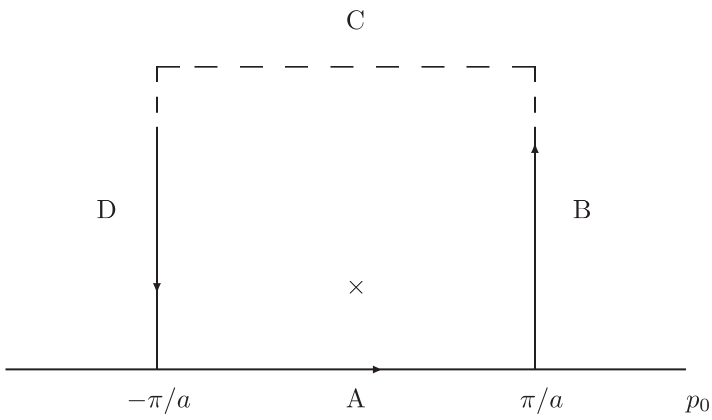


Fig. 3.1

while for $x_0 > 0$, the integral over C vanishes, $\int_C (...) = 0$. Therefore we end up with the relation

$$\int_A (...) = 2\pi i \sum_{poles} \text{Residues}. \tag{3.22}$$

Single poles occur at the solutions of the equation:

$$\hat{p}^2 + m^2 = 0 \Rightarrow p_0 = \pm i\omega(\mathbf{p}) \left(mod \ \frac{2\pi}{a} \right), \tag{3.23}$$

which are purely complex in the BZ. $\omega(\mathbf{p})$ is a real number satisfying:

$$\cosh \omega(\mathbf{p})a = 1 + \frac{a^2}{2} \left(m_0^2 + \frac{4}{a^2} \sum_{i=1}^{3} \sin^2 \frac{p_i a}{2} \right). \tag{3.24}$$

There is only one pole within the closed contour, $p_0 = +i\omega(\mathbf{p})$ with residue

$$\text{Residue}[p_0 = +i\omega(\mathbf{p})] = \frac{1}{2\bar{\omega}(\mathbf{p})}, \qquad \bar{\omega}(\mathbf{p}) \equiv \frac{1}{a} \sinh\left(\omega(\mathbf{p})a\right) \tag{3.25}$$

and therefore

$$\langle \phi(x)\phi(0) \rangle = \int_A (...) = \frac{1}{2\bar{\omega}(\mathbf{k})} e^{-\omega(\mathbf{k})x_0} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.26}$$

We indeed recover the expected behaviour if we identify the one-particle energies $E_{\mathbf{p}}(\alpha) \to \omega(\mathbf{p})$, while the matrix elements

$$|\langle 0|\hat{\phi}(0)|\alpha\rangle| \to \sqrt{\frac{\omega(\mathbf{p})}{\bar{\omega}(\mathbf{p})}}. \tag{3.27}$$

Had we started with the canonical quantization of the free lattice scalar field we would have arrived at the same result.

The continuum limit $a \to 0$ can be readily obtained:

$$\lim_{a\to 0} \omega(\mathbf{p}) = \lim_{a\to 0} \bar{\omega}(\mathbf{p}) = \sqrt{m_0^2 + \mathbf{p}^2} + \mathcal{O}(a^2), \tag{3.28}$$

as expected.

Exercise 2.1 Show that the free scalar Euclidean propagator in a periodic box of extent T and L is given by

$$\langle \phi(x)\phi(0)\rangle = L^{-3} \sum_{\mathbf{p}} \frac{\cosh\left[E_{\mathbf{p}}(\frac{T}{2} - x_0)\right]}{2E_{\mathbf{p}} \sinh\left(\frac{T}{2}E_{\mathbf{p}}\right)}, \quad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2} \tag{3.29}$$

Use:

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + \alpha^2} = -\frac{1}{2\alpha^2} + \frac{\pi}{2\alpha} \frac{\cosh\left(\alpha(\pi - x)\right)}{\sinh(\alpha\pi)}, \quad 0 \le x \le 2\pi. \tag{3.30}$$

Show that in the infinite volume limit, the correct KL representation is obtained.

3.2 Interacting lattice scalar theory

When $\lambda \neq 0$ the theory cannot be solved analytically, however one can rigorously prove the fundamental property of unitarity or the existence and uniqueness of the Hilbert space representation. This can be done by the following steps:

- Identification of a transfer operator $\hat{T}$ and field operator $\hat{\phi}$ such that

$$\langle \phi(x_1)...\phi(x_n)\rangle = \lim_{T\to\infty} \frac{\mathrm{Tr}\left[\hat{T}^{(T/2 - x_1^0)/a}\hat{\phi}(0,\mathbf{x}_1)\hat{T}^{(x_1^0 - x_2^0)/a}\hat{\phi}(0,\mathbf{x}_2)...\hat{T}^{(T/2 + x_n^0)/a}\right]}{\mathrm{Tr}[\hat{T}^{T/a}]}. \tag{3.31}$$

  In general, it takes some guesswork to identify the transfer operator. In this case, it is easy to see that it may be chosen as that of eq. (2.52) by simply substituting the continuum derivatives by the discrete ones, and the integrals over space by sums.

- Prove that $\hat{T}$ is strictly positive (see exercise). For any $|\Psi\rangle$:

$$\langle\Psi|\hat{T}|\Psi\rangle > 0, \qquad \langle\Psi|\Psi\rangle = 1. \tag{3.32}$$

- Prove that $\hat{T}$ and $\hat{\phi}$ are unique (up to unitary transformations). This is the content of the *reconstruction theorem* (Streater and Wightman, 1964).

All these conditions imply that the quantum Hamiltonian $\hat{H} \equiv -\frac{1}{a}\ln\hat{T}$ is self-adjoint and unique.

Alternatively one can invoke the *Osterwalder-Schrader reflection positivity condition* which ensures unitarity as a result of a property of Euclidean correlation functions (i.e. without the need to identify the Hilbert space transfer operator). The *time-reflection positivity* condition is the following. Let O be any product of the classical fields at positive times:

$$O(x_1^0, ..., x_n^0) = \phi(x_1)....\phi(x_n), \quad x_i^0 > 0. \tag{3.33}$$

We define the operation, $\theta[...]$ of time reflection as

$$\theta\left[O(x_1^0, ..., x_n^0)\right] = O(-x_1^0, ..., -x_n^0). \tag{3.34}$$

If for any such polynomial it is true that

$$\langle\theta\left[O^\dagger\right]O\rangle \geq 0, \tag{3.35}$$

we say that the theory has reflection positivity, which ensures (Osterwalder and Schrader, 1973; Osterwalder and Schrader, 1975)

- positivity of the scalar product in Hilbert space
- positivity of $\hat{T}^2$, which is the operator that generates times translations by $2a$ and therefore a Hermitian Hamiltonian $\hat{H} = -\frac{1}{2a}\ln\hat{T}^2$

Exercise 2.2 Prove the unitarity of the lattice scalar model by

a) showing that the transfer matrix is a positive operator
b) showing that the lattice formulation has the property of reflection positivity. To show this, show that the action can be written as

$$S = S_+ + S_0 + S_-, \tag{3.36}$$

where S_0 depends only on the fields at $x_0 = 0$, S_+ on the fields at $x_0 > 0$ and S_- on the fields at $x_0 < 0$. Show that

$$\theta(S_+) = S_-. \tag{3.37}$$

Rewrite the correlation function of eq. (3.35) in a manifestly positive way.

Table 3.1 One loop contributions to $\Gamma^{(2)}$ and $\Gamma^{(4)}$

3.3 Lattice Perturbation Theory

Deriving the perturbative expansion and Feynman rules from the lattice theory is completely analogous to the continuum. We treat

$$S^{(1)} = a^4 \sum_x \frac{\lambda}{4!} \phi(x)^4, \tag{3.38}$$

as a perturbation in the path integral, eq. (2.82).

The Feynman rules for this theory are just like those in the continuum with the propagator substituted by the lattice one of eq. (3.18), while the vertex is the same: it connects four scalar lines with strength $-\lambda$. The combinatorial factors coming from Wick contractions are also just like in the continuum.

Let's consider the one-loop corrections to the two and four vertex functions (Figs. 3.1):

$$\begin{aligned}
\Gamma^{(2)}(p,-p) &= -(\hat{p}^2 + m_0^2) - \frac{\lambda}{2}\int_{BZ} \frac{d^4k}{(2\pi)^4} \frac{1}{\hat{k}^2 + m_0^2} \equiv -(\hat{p}^2 + m_0^2) - \frac{\lambda}{2} I_1(a, m_0) \\
\Gamma^{(4)}(p_1,p_2,p_3,p_4) &= -\lambda + \left(\frac{\lambda^2}{2} \int_{BZ} \frac{d^4k}{(2\pi)^4} \frac{1}{(\hat{k}^2 + m_0^2)(\widehat{k + p_1 + p_2})^2 + m_0^2} + \text{perm} \right) \\
&\equiv -\lambda + \frac{\lambda^2}{2} \left(I_2(a, m_0, p_1 + p_2) + \text{perm.} \right).
\end{aligned} \tag{3.39}$$

As in the previous example, all Feynman graphs satisfy the following properties in momentum space:

- periodic functions of all momenta with periodicity $2\pi/a$ in each momentum direction
- loop momenta are integrated only in the BZ and are therefore finite

On the lattice, divergences are expected when we try to approach the continuum limit $a \to 0$. The expectation from perturbative renormalizability is that a continuum limit can be taken provided a tuning of m, λ and the field normalization are performed. It is easy to check that this is indeed the case at the one loop order.

The $\Gamma^{(2)}$ above does not have a finite continuum limit since

$$I_1(a, m_0) = \int_{BZ} \frac{d^4k}{(2\pi)^4} \frac{1}{\hat{k}^2 + m_0^2} = \frac{1}{a^2} F(m_0 a), \tag{3.40}$$

and the function $F(x)$ does not vanish for small x:

$$F(0) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\sum_\mu (\sin k_\mu/2)^2} = 0.154933... \tag{3.41}$$

The first derivative is however not defined at $m_0 a = 0$, because it has a logarithmic divergence. Isolating this divergence, we find:

$$I_1(a, m_0) = \frac{1}{a^2} F(m_0 a) = \frac{F(0)}{a^2} - m_0^2 \left(-\frac{1}{16\pi^2} \ln(m_0 a)^2 + C + \mathcal{O}(m_0 a)^2 \right), \tag{3.42}$$

where $C = 0.030345755....$

In this simple example, it is easy to show that the divergent constant of eq. (3.40) can be reabsorbed by a redefinition of m_0^2

$$\Gamma^{(2)}(p, -p) = -(\hat{p}^2 + m_0^2) - \frac{\lambda}{2} I_1(a, m_0) \equiv -(\hat{p}^2 + m_R^2). \tag{3.43}$$

Similarly if we consider the $\Gamma^{(4)}$ vertex function we find that the integral I_2 is divergent. If we consider the Taylor expansion with respect to external momenta, we find that the divergence is present only in the leading term (i.e. at zero external momenta):

$$\begin{aligned} I_2(a, m_0, 0) &= \int_{BZ} \frac{d^4k}{(2\pi)^4} \frac{1}{(\hat{k}^2 + m_0^2)^2} = -\frac{d}{dm_0^2} I_1(a, m_0) \\ &= C - \frac{1}{16\pi^2} (\ln(m_0 a)^2 - 1) + \mathcal{O}(a^2), \end{aligned} \tag{3.44}$$

therefore the corresponding divergence can be reabsorbed in λ:

$$\Gamma^{(4)}(0,0,0,0) = -\lambda + \frac{3\lambda^2}{2} I_2(a, m_0, 0) \equiv -\lambda_R. \tag{3.45}$$

The renormalized quantities are therefore

$$\begin{aligned} m_R^2 &= m_0^2 + \frac{\lambda}{2} \left(\frac{F(0)}{a^2} + \frac{m_0^2}{16\pi^2} \ln(m_0 a)^2 - C m_0^2 \right), \\ \lambda_R &= \lambda + \frac{3\lambda^2}{2} \left(-C + \frac{1}{16\pi^2} (\ln(m_0 a)^2 + 1) \right). \end{aligned} \tag{3.46}$$

This way of redefining the renormalized couplings corresponds to the usual mass-shell scheme:

$$\Gamma^{(2)}(0,0) = -m_R^2, \quad \left. \frac{d\Gamma^{(2)}(p,-p)}{dp^2} \right|_{p=0} = 1, \quad \Gamma^{(4)}(0,0,0,0) = -\lambda_R. \tag{3.47}$$

That this must hold to all orders of perturbation theory requires a non-trivial theorem known as the *Reisz power counting theorem* (Reisz, 1988). It is the analog of the continuum one, and permits to carry the BPHZ recursive renormalization procedure over to the lattice regularization. This has been discussed in P. Weisz's lectures (Weisz, 2009).

3.4 Callan-Symanzik equations. Beta functions.

We have already discussed the renormalization group and why approaching the continuum limit can be seen as a flow in the space of couplings. As we have seen above, the continuum scalar theory that we are trying to describe has one relevant direction, m_0, and one marginal one λ. As we approach the continuum limit, the quantities of eq. (3.46) must be tunned.

In the Wilsonian RG we have seen that as we approach the continuum limit, the effective couplings change smoothly in a way that is locally determined by the effective couplings themselves. We can therefore derive a differential equation to describe this change. These are the famous *Callan-Symanzik equations*. Let us consider a fixed λ and let us see how λ_R changes with a. We tune m so that m_R is fixed to the physical mass as we approach the continuum limit. Differenciating the second eq. (3.46) we find at leading order in the perturbative expansion:

$$\beta(\lambda_R) \equiv a\frac{d\lambda_R}{da}\bigg|_{\lambda} = \frac{3}{(16\pi^2)}\lambda^2 + \mathcal{O}(\lambda^3) = \frac{3}{(16\pi^2)}\lambda_R^2 + \mathcal{O}(\lambda_R^3). \tag{3.48}$$

This is the Callan-Symanzik beta function.

This function can be computed to higher orders, for instance the two loop result is

$$\beta(\lambda) = \beta_0\lambda^2 + \beta_1\lambda^3 + \ldots \tag{3.49}$$

and the coefficients β_0 and β_1 can be shown to be universal (do not depend on the regularization scheme):

$$\beta_0 = \frac{3}{16\pi^2}, \quad \beta_1 = -\frac{17}{3(16\pi^2)^2}. \tag{3.50}$$

The equation can be integrated to give

$$a = Ce^{-1/(\beta_0\lambda_R)}\lambda_R^{-\beta_1/\beta_0^2}\left(1 + \mathcal{O}(\lambda_R)\right), \tag{3.51}$$

where C is some integration constant that must be determined from initial conditions. This equation shows that as we approach the continuum limit

$$\lim_{a\to 0}\lambda_R(a)\bigg|_{\lambda} \sim \lim_{a\to 0}\frac{1}{\ln a} = 0, \tag{3.52}$$

so the continuum theory has a vanishing renormalized coupling, i.e. it is *trivial*. Unfortunately this argument is not a sufficient proof of triviality, because it is based on perturbation theory. The question is of course if one could use the lattice formulation to go beyond.

3.5 Triviality in lattice $\lambda\phi^4$ (and in the SM)

The Higgs sector of the Standard Model is a multicomponent $\lambda\phi^4$ theory, with a continuous global symmetry that is spontaneously broken. The β function of the bare coupling, λ, has the same properties as in the single scalar case: the renormalized coupling decreases as we approach the continuum limit at fixed bare coupling. Therefore one should worry that actually this theory cannot be defined without a cutoff, or if one does then it is a trivial theory $\lambda_R = 0$, which would not be in agreement with phenomenology. In particular the Higgs mass is related to the renormalized coupling in the following way:

$$\frac{m_H^2}{v^2} = \frac{\lambda_R}{3}. \tag{3.53}$$

Therefore taking the cutoff to ∞ would imply in particular a massless Higgs.

If we do not remove the cutoff, we can try to maximize the value of λ_R modifying λ in all its possible range: $\lambda \in [0, \infty)$. For example, we could lower the cutoff as much as possible:

$$\frac{\Lambda}{m_H} \geq 2, \tag{3.54}$$

so that the cutoff is higher than two times the Higgs mass (otherwise the SM would not make sense not even as an effective theory). Such a condition implies an upper bound on λ_R:

$$\lambda_R \leq \lambda_R^{max}, \tag{3.55}$$

and therefore an upper bound to the Higgs mass, according to eq. (3.53).

This problem has a very definite answer in the lattice regularization and it was studied extensively in the late eighties. The picture that emerged from numerical studies as well as analytically is that indeed the only IR fixed-point in the discretized scalar theories is the trivial one and the theory is trivial in the continuum limit.

The method followed by Lüscher-Weisz (Lüscher and Weisz, 1988; Lüscher and Weisz, 1989) can be summarized as follows. The (m_0, λ) space can be mapped to the $(\kappa, \bar{\lambda})$ space, where the original lattice action is written as

$$S = a^4 \sum_x \left[\phi(x)^2 + \bar{\lambda}(\phi(x)^2 - 1)^2 - \kappa \sum_\mu \left(\phi(x)\phi(x+\hat{\mu}) + \phi(x)\phi(x-\hat{\mu})\right) \right], \tag{3.56}$$

after the change of variables

$$\phi(x) \to \sqrt{2\kappa}\phi(x) \quad a^2 m_0^2 \to \frac{1-2\bar{\lambda}}{\kappa} - 8 \quad \lambda \to \frac{6\bar{\lambda}}{\kappa^2}. \tag{3.57}$$

There is a critical line $\kappa_c(\bar{\lambda})$, where the mass vanishes, where the continuum limit should lie. For values of κ sufficiently far from this line, the so-called hopping parameter expansion (or high temperature expansion), a Taylor series in κ, is convergent. The strategy to study the triviality of the theory follows the following steps:

- Use the hopping parameter expansion or high temperature expansion to compute m_R and λ_R (as defined by some renormalization prescription such as the onshell one, defined above) in a region of κ not too close to κ_c. The fact that the series has been computed to very high order, allows to control very well the truncation error for values of λ_R that are already in the perturbative domain

$$m_R a = \frac{1}{\sqrt{\kappa}} \sum_n \alpha_n(\bar{\lambda})\kappa^n,$$
$$\lambda_R = \sum_n \beta_n(\bar{\lambda})\kappa^n. \tag{3.58}$$

For $m_R a \sim 0.5$, we are sufficiently far from the critical line to have an accurate description, while λ_R is rather small. Note that if $m_R a \geq 0.5$ means the cutoff is of the order of the mass.

- Solve the perturbative Callan-Symanzik equations for the renormalized coupling in order to approach the critical line with initial conditions given by the results of the hopping expansion. Since the initial λ_R is small enough and it gets smaller as we approach the continuum limit, the procedure is under control.

In this way, Lüscher-Weisz could map the lines of constant (m_R, λ_R) as the cutoff changes. As $m_R a$ decreases along these lines, we get closer to $\bar{\lambda} = \infty$, which is the furthest we can get, so one can read the bound on λ_R by considering this value of the bare coupling. The result can be plotted in the renormalized plane $(m_R a)^{-1}$ vs m_R/v_R at $\bar{\lambda} = \infty$ as shown in Figure 3.2. At $m_R a \sim 0.5$ we can read the value of m_R/v_R, resulting in the limit (Lüscher and Weisz, 1988)

$$m_H \leq 630GeV, \tag{3.59}$$

for the $O(4)$ model. These results agree with the numerical studies e.g. (Montvay, Münster and Wolff, 1988; Hasenfratz *et al.*, 1987), therefore the issue is settled, to the extent that neglecting fermion and gauge field effects in the SM is a good approximation. For a review of the triviality problem see (Callaway, 1988).

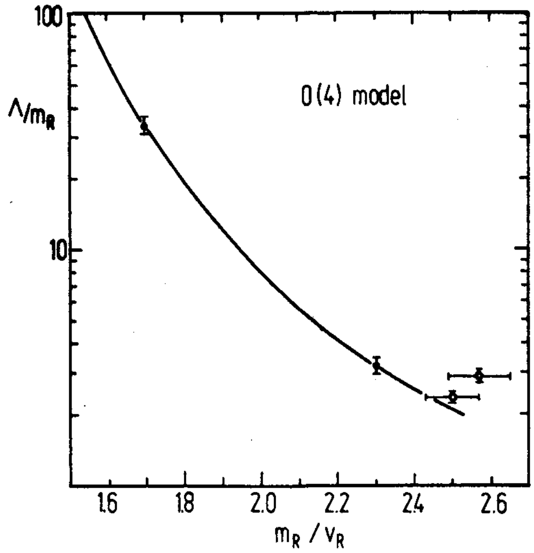


Fig. 3.2 Value of $m_R a$ as a function of m_R/v in the $O(4)$ scalar model as obtained by Lüscher and Weisz (Lüscher and Weisz, 1988).

4
Free fermions on the lattice

The Fock space of fermions can be reconstructed from the vacuum acting with creation and annihilation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$, satisfying the following canonical anticommutation relations

$$\{\hat{a}_k, \hat{a}_l\} = \{\hat{a}_k^\dagger, \hat{a}_l^\dagger\} = 0, \quad \{\hat{a}_k, \hat{a}_l^\dagger\} = \delta_{kl}. \tag{4.1}$$

An arbitrary normalized state $|\psi\rangle$ can be written

$$|\psi\rangle = \sum_p \frac{1}{p!} \psi_{k_1,...k_p} \hat{a}_{k_1}^\dagger ....\hat{a}_{k_p}^\dagger |0\rangle. \tag{4.2}$$

In the functional formalism (Berezin, 1966), the fermion classical fields are elements of a Grassmann algebra. The generators are a set of anticommuting variables $c_1, ...c_n$ and $\bar{c}_1, ...\bar{c}_n$, with the following anticommutation properties:

$$\{c_i, c_j\} = \{c_i, \bar{c}_j\} = \{\bar{c}_i, \bar{c}_j\} = 0, \tag{4.3}$$

which imply that $c_i^n = 0, n \geq 2$. The elements of the algebra are elements of the form

$$X_{n_1,...,n_n,m_1,...,m_n} = c_1^{n_1}....\bar{c}_n^{m_n}, \quad n_i, m_i \in \{0, 1\}. \tag{4.4}$$

Any function of the Grassmann variables can be represented by a series expansion:

$$f(c, \bar{c}) = \sum_{n_i, m_i} f_{n_1....m_n} X_{n_1,...m_n}. \tag{4.5}$$

We can define the integral over all Grassmann variables as:

$$\int d\bar{c}\; dc\; f(c, \bar{c}) = f_{111...1}. \tag{4.6}$$

Note that this implies in particular

$$\int dc_i = 0, \quad \int dc_i c_i = 1. \tag{4.7}$$

In defining the partition function for fermions we will find integrals of the form

$$\mathcal{Z}_F \equiv \int d\bar{c}dc\; \exp\left\{-\sum_{i,j} \bar{c}_i M_{ij} c_j\right\} = \frac{(-1)^n}{n!} \int d\bar{c}dc \left(\sum_{i;j} \bar{c}_i M_{ij} c_j\right)^n =$$

$$\frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \sum_{i_1,\ldots i_n; j_1,\ldots j_n} \epsilon_{i_1,\ldots i_n} \epsilon_{j_1,\ldots j_n} M_{i_1 j_1} \ldots\ldots M_{i_n j_n} = (-1)^{\frac{n(n-1)}{2}} \det(M). \tag{4.8}$$

And for correlation functions involving fermions we need integrals of the form:

$$\begin{aligned}\langle c_{k_1} \bar{c}_{l_1} \ldots . c_{k_m} \bar{c}_{l_m} \rangle_F &\equiv \mathcal{Z}_F{}^{-1} \int d\bar{c} dc \; c_{k_1} \bar{c}_{l_1} \ldots . c_{k_m} \bar{c}_{l_m} \; \exp\left\{ -\sum_{i,j} \bar{c}_i M_{ij} c_j \right\} \\ &= \sum_{perm} (-1)^{\sigma(perm)} \langle c_{k_1} \bar{c}_{l_1} \rangle_F \ldots \langle c_{k_m} \bar{c}_{l_m} \rangle_F, \end{aligned} \tag{4.9}$$

where each contraction is between a c and a $\bar{c}$ variable

$$\langle c_{k_m} \bar{c}_{l_m} \rangle_F = (M^{-1})_{k_m l_m}. \tag{4.10}$$

The Euclidean action for free Dirac fermions of mass m is given by

$$S[\psi, \bar{\psi}] = \int d^4x \; \frac{1}{2} \left[\bar{\psi}(x) \gamma_\mu \partial_\mu \psi(x) - \partial_\mu \bar{\psi}(x) \gamma_\mu \psi(x) \right] + m \bar{\psi}(x) \psi(x), \tag{4.11}$$

where we can choose the *chiral representation* of the γ matrices:

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^\dagger & 0 \end{pmatrix}, \tag{4.12}$$

and the 2×2 matrices are taken to be:

$$e_0 \equiv -I, \quad e_k \equiv -i\sigma_k, \tag{4.13}$$

where σ_k are the Pauli matrices. It is easy to check the following properties

$$\gamma_\mu^\dagger = \gamma_\mu \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{4.14}$$

We also define

$$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3, \tag{4.15}$$

satisfying

$$\gamma_5^\dagger = \gamma_5, \quad \gamma_5^2 = 1. \tag{4.16}$$

The mapping of a single Dirac fermion on the Grassmann algebra is therefore

$$\{c_1, \ldots, c_n; \bar{c}_1, \ldots \bar{c}_n\} \to \{\psi_\alpha(x); \bar{\psi}_\alpha(x)\}_x^{\alpha=1,..4} \tag{4.17}$$

The number of c and $\bar{c}$ Grassmann variables to represent a general fermion is therefore $4 \times N_{flavour} \times N_{color} \times$ space-time points. The partition function is

$$\mathcal{Z}_F = \int d\bar{\psi} d\psi e^{-S[\psi,\bar{\psi}]}. \tag{4.18}$$

As in the scalar theory, the propagator of the theory gives us information on the one-particle asymptotic states of the theory via the Källen-Lehmann representation of the propagator. At large Euclidean time, the fermion propagator should behave as

$$\langle 0|\psi(x)\bar{\psi}(0)|0\rangle_F\big|_{x_0>0} = \sum_\alpha \int \frac{d^3p}{(2\pi)^3}\, Z_\alpha^2 \, \frac{i\gamma_\mu p_\mu - m}{2ip_0}\bigg|_{p_0=i\sqrt{m_\alpha^2+\mathbf{p}^2}} e^{-E_{\mathbf{p}}(\alpha)x_0} e^{i\mathbf{px}}, \tag{4.19}$$

with $E_{\mathbf{p}}(\alpha) = \sqrt{m_\alpha^2 + \mathbf{p}^2}$. This is the KL representation for fermions that can be derived analogously to the scalar case.

4.1 Naive fermions

Let us now try to discretize the Euclidean action in the same way we did for the scalar fields. The fields are now defined at the lattice points only and the derivatives are substituted by their discrete versions. We find therefore the so-called *naive fermion action*:

$$S[\psi,\bar{\psi}] = a^4 \sum_{x,\alpha,\mu} \bar{\psi}_\alpha(x) \left[\frac{1}{2}(\hat{\partial}_\mu + \hat{\partial}_\mu^*) + m\right] \psi_\alpha(x) = a^4 \sum_{x,y} \bar{\psi}_\alpha(x) K_{xy}^{\alpha\beta} \psi_\beta(y), \tag{4.20}$$

where

$$K_{xy}^{\alpha\beta} \equiv \sum_\mu \frac{1}{2a} (\gamma_\mu)_{\alpha\beta} \left(\delta_{yx+a\hat{\mu}} - \delta_{yx-a\hat{\mu}}\right) + m\delta_{\alpha\beta}\delta_{xy}. \tag{4.21}$$

We can understand the particle interpretation of this theory by studying the Källen-Lehmann representation of the propagator. According to the Grassmann integration rules, it is given by

$$\langle \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle_F = \frac{1}{a^4}\left(K^{-1}\right)_{xy}^{\alpha\beta}. \tag{4.22}$$

The propagator can be easily computed in momentum space:

$$K_{pq}^{\alpha\beta} = a^4 \left[\sum_\mu \frac{i}{a}\gamma_\mu \sin(q_\mu a) + m\right]_{\alpha\beta} (2\pi)^4 \delta_P(p+q), \tag{4.23}$$

so that

$$\langle \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle_F = \int_{BZ} \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{\sum_\mu i\gamma_\mu \frac{\sin(p_\mu a)}{a} + m}. \tag{4.24}$$

As we did in the case of the scalar field, we first perform the integration over p_0. We deform the integration into the complex plane depicted in Fig. 3.1. The integral can then be written as a sum of residues of single poles in the band $|\mathrm{Re}\ p_0| \leq \pi/a$ and $\mathrm{Im} p_0 \geq 0$. Contrary to the scalar case, we find two poles in this region, satisfying:

$$e^{ip_0 a} = \pm e^{-\omega_{\mathbf{p}} a} \equiv \pm \left(\sqrt{1 + M_{\mathbf{p}}^2} - M_{\mathbf{p}}\right) \tag{4.25}$$

with

$$M_{\mathbf{p}}^2 \equiv m^2 a^2 + \sum_{k=1}^{3} \sin(p_k a)^2. \tag{4.26}$$

The integral can be easily performed and gives:

$$\begin{aligned}\langle \psi_\alpha(x)\bar{\psi}_\beta(0)\rangle_F = &\int \frac{d^3p}{(2\pi)^3} \frac{e^{i\mathbf{p}\mathbf{x}} e^{-\omega_{\mathbf{p}} x_0}}{\sinh(2\ \omega_{\mathbf{p}} a)} \left[\left(\gamma_0 \sinh \omega_{\mathbf{p}} a - i \sum_k \gamma_k \sin p_k a + ma\right)\right. \\ &+ \left.(-1)^{x_0/a} \left(-\gamma_0 \sinh \omega_{\mathbf{p}} a - i \sum_k \gamma_k \sin p_k a + ma\right)\right]. \end{aligned} \tag{4.27}$$

Two new features appear with respect to the scalar case:

- there are two terms in the sum with the same energy, $\omega_{\mathbf{p}}$, but different residue
- the energy, $\omega_{\mathbf{p}}$, as a function of the spatial momenta in one direction (the others are set to zero) is shown in Fig. 4.1. There are two different minima in the BZ (the one at $-\pi$ is the same as that at π by periodicity). More generically, we find 2^3 minima at

$$p_k = \bar{p}_k \equiv n_k \frac{\pi}{a} \quad n_k = 0, 1. \tag{4.28}$$

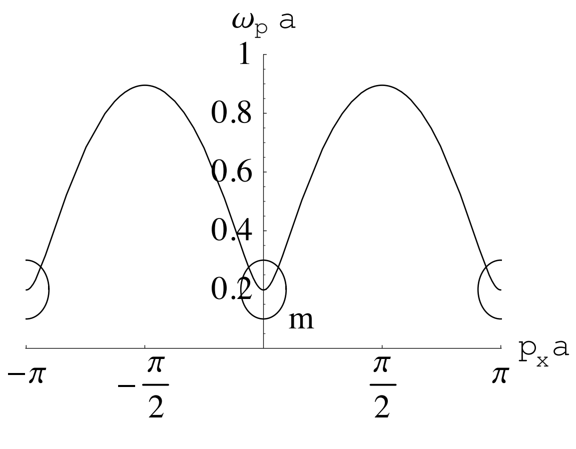


Fig. 4.1 $\omega_{\mathbf{p}}$ as a function of $p_x a$ for $p_y = p_z = 0$.

As we approach the continuum limit:

$$\lim_{a \to 0} \quad \omega_{\mathbf{p}}\big|_{p_k = n_k \pi/a} = m. \tag{4.29}$$

Therefore the minima correspond to the same energy.

Near the continuum limit, it is justified to consider the contribution near these momenta, so let us consider the expansion around them:

$$p_j = \bar{p}_j^{(i)} + k_j, \quad k_j a \ll 1, \tag{4.30}$$

where $j = 1, ..., 2^3$. It is easy to see that

$$\begin{gathered} \sinh 2\omega_{\mathbf{p}} a \simeq a k_0 + O(a^2) \quad \sinh \omega_{\mathbf{p}} a \simeq a k_0 + O(a^2), \\ \sin p_j a \simeq \cos(\bar{p}^{(i)} a) k_j a + O(a^2), \quad k_0 \equiv \sqrt{m^2 + \mathbf{k}^2} \end{gathered} \tag{4.31}$$

Putting it all together

$$\begin{aligned} &\sum_{i=1}^{8} e^{i\mathbf{p}^{(i)}\mathbf{x}} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\mathbf{x}} e^{-\omega_{\mathbf{p}} t}}{2k_0} \left[\left(\gamma_0 k_0 - i \sum_j \gamma_j \cos(\bar{p}_j^{(i)} a) k_j + m \right) \right. \\ &+ \left. (-1)^{t/a} \left(-\gamma_0 k_0 - i \sum_j \cos(\bar{p}_j^{(i)} a) \gamma_j k_j + m \right) \right] \\ &= \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)} x} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\mathbf{x}} e^{-\omega_{\mathbf{p}} t}}{2k_0} \left[\gamma_0 \cos(\bar{p}_0^{(\alpha)} a) k_0 - i \sum_j \gamma_j \cos(\bar{p}_j^{(\alpha)} a) k_j + m \right], \end{aligned} \tag{4.32}$$

where we have used the fact that the second term can be written in the same form as the first, corresponding to a different temporal momenta $\bar{p}_0 = \pi/a$. The 16 terms now correspond to

$$\bar{p}_\mu = (n_0, n_1, n_2, n_3) \frac{\pi}{a}, \quad n_\mu = 0, 1. \tag{4.33}$$

We can find unitarity operators S_a such that

$$S_\alpha \gamma_\mu S_\alpha^\dagger = \gamma_\mu \cos(\bar{p}_\mu^{(\alpha)} a). \tag{4.34}$$

For example

$$S_\alpha = \prod_\mu (i\gamma_\mu \gamma_5)^{n_\mu^{(\alpha)}}, \tag{4.35}$$

satisfies this property. Therefore we can write the continuum limit as

$$\sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)} x} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\mathbf{x}} e^{-\omega_{\mathbf{p}} t}}{2k_0} S_\alpha \left[\left(\gamma_0 k_0 - i \sum_k \gamma_k k_k + m \right) \right] S_\alpha^{-1}. \tag{4.36}$$

We can now recognize in each term the contribution of a relativistic fermion in the continuum, eqs. (4.19), since S_α is just a similarity transformation: an equivalent representation of the γ matrices.

Summarizing, we have found that the continuum limit contains 16 relativistic free fermions instead of 1. This is the famous *doubling problem* (Wilson, 1975; Susskind, 1977).

4.2 Doubling and chiral symmetry

There is a deep connection between the doubling problem and the difficulty to regularize chirality (Nielsen and Ninomiya, 1981).

It is well-known that in the absence of a mass term, the free fermion action has a global symmetry under chiral rotations:

$$\psi(x) \to e^{i\alpha\gamma_5}\psi(x). \tag{4.37}$$

The naive discretization we just considered also has an exact global symmetry of this form. The invariance under chiral rotations implies that the Dirac spinor representation is actually reducible to its chiral components. Therefore in the continuum we can consider a free Weyl fermion as the left or right chiral component that we can define by applying a projector on the Dirac field

$$\psi_L \equiv \frac{1-\gamma_5}{2}\psi, \quad \psi_R \equiv \frac{1+\gamma_5}{2}\psi. \tag{4.38}$$

Let us see what happens with the doublers when we naively discretize the action for a Weyl fermion. The naive propagator, eq. (4.24), is (for $m=0$):

$$\langle \psi_L(x)\bar{\psi}_L(0)\rangle_F = \int_{BZ} \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{ia^{-1}\gamma_\mu \sin p_\mu a}\left(\frac{1-\gamma_5}{2}\right) =$$

$$\sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}x}\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\mathbf{x}}e^{-\omega_{\mathbf{p}}t}}{2k_0}S_\alpha\left[\left(\gamma_0 k_0 - i\sum_k \gamma_k k_k\right)S_\alpha^{-1}\left(\frac{1-\gamma_5}{2}\right)\right.$$

$$= \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}x}\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\mathbf{x}}e^{-\omega_{\mathbf{p}}t}}{2k_0}S_\alpha\left[\left(\gamma_0 k_0 - i\sum_k \gamma_k k_k\right)\left(\frac{1-\epsilon^a\gamma_5}{2}\right)\right]S_\alpha^{-1}, \tag{4.39}$$

where $\epsilon^\alpha = (-1)^{\sum_\mu n_\mu^{(\alpha)}}$. Therefore each of the doublers contribute either a left-handed relativistic Weyl fermion for $\epsilon^\alpha = 1$ or a right-handed one $\epsilon^\alpha = -1$ in the continuum. It turns out that the number of right and left movers is the same!

$$\text{Left}: 1+6+1 = 8, \tag{4.40}$$

$$\text{Right}: 4+4 = 8. \tag{4.41}$$

This result can be generalized to rather arbitrary forms of the fermionic action. It is the content of the famous *Nielsen-Ninomiya theorem* (Nielsen and Ninomiya, 1981). In its Euclidean version, the theorem considers actions of the form

$$S_F = a^4 \sum_{x,y} \bar{\psi}(x)\gamma_\mu F_\mu(x-y)(1-\gamma_5)\psi(y), \tag{4.42}$$

satisfying the following properties:

- Action quadratic in the fermion fields
- Invariant under lattice translations (i.e. diagonal in momentum space)
- Local (smooth Fourier transform)
- Hermitian action: $F_\mu(x)^* = -F_\mu(x)$ (implies a real Fourier transform of F_μ field)

We also assume that the function F_μ has some isolated zeros (in order to have a continuum limit). Let us call $\bar{p}^\alpha$ the zeros of $F_\mu(p)$. Sufficiently close we can approximate

$$F_\mu(p) \simeq M^{(\alpha)}_{\mu\nu}(p - \bar{p}^\alpha)_\nu + ... = M^{(\alpha)}_{\mu\nu} k^{(\alpha)}_\nu + ..., \tag{4.43}$$

where $M^{(\alpha)}_{\mu\nu}$ is a real matrix that can be decomposed in general as

$$M^{(\alpha)}_{\mu\nu} = O^{(\alpha)}_{\mu\rho} S^{(\alpha)}_{\rho\nu}, \tag{4.44}$$

where O^α is an orthogonal matrix and S^α is a positive real symmetric matrix. The orthogonal matrix can be reabsorbed in a unitarity rotation of the fields for the following reason. Consider a rotation in $d+1$ (with d even) Euclidean space which acts in the first d coordinates as O and in the last coordinate it multiplies by $\det^{-1} O = \pm 1$. Such a rotation therefore belongs to $SO(d+1)$. The spinor representation of such rotations are the $d+1$ γ matrices: (γ_μ, γ_5). There must exist therefore a unitary matrix that implements the rotation in the spinor representation such that

$$\Lambda^{(\alpha)} \gamma_\nu {\Lambda^{(\alpha)}}^{-1} = O^{(\alpha)}_{\mu\nu} \gamma_\mu \quad \Lambda^{(\alpha)} \gamma_5 {\Lambda^{(\alpha)}}^{-1} = \det{}^{-1} O^{(\alpha)} \gamma_5. \tag{4.45}$$

Therefore we can rewrite the action as

$$\sum_\alpha \int \frac{d^4 k^{(\alpha)}}{(2\pi)^4} \bar{\psi}(-k^{(\alpha)}) \Lambda^{(\alpha)} \gamma_\rho S^{(\alpha)}_{\rho\nu} k^{(\alpha)}_\nu (1 - \det O^{(\alpha)} \gamma_5) {\Lambda^{(\alpha)}}^{-1} \psi(k^{(\alpha)}). \tag{4.46}$$

The real positive matrix $S^{(\alpha)}$ is harmless and can be reabsorbed in a rescaling of the momentum. However we see that there are left-movers and right-movers depending on the sign of $\det O^{(\alpha)}$. A theorem by Poincaré-Hopf states that $\sum_\alpha \det O^{(\alpha)}$ is the Euler characteristic of the manifold on which the vector $F_\mu(p)$ is defined. It is zero for the Brillouin zone (which is topologically a four-torus). Therefore there must be as many zeros with $\det O^{(\alpha)} = 1$ as those with $\det O^{(\alpha)} = -1$. In particular this implies the number of zeros cannot be one!

Intuitively, this is a generalization of a simpler version of the theorem for one dimensional functions: a smooth and periodic function that crosses zero must do it an even number of times with opposite signs of the derivatives at the zeros.

Not surprisingly the easiest way to get rid of doublers is to break chiral symmetry. This is Wilson's solution to the doubling problem (Wilson, 1975).

4.3 Wilson fermions

K. Wilson proposed to add to the naive action the following term

$$\Delta_W S = -a^4 \sum_x \bar{\psi}(x) \frac{ra}{2} \hat{\partial}^*_\mu \hat{\partial}_\mu \psi(x). \tag{4.47}$$

where r is some arbitrary constant of $O(1)$. Note that this term does break explicitely chiral symmetry since it is like a momentum-dependent mass term. It is easy to see that the propagator in momentum space is modified to

$$\langle\psi_\alpha(x)\bar{\psi}_\beta(y)\rangle_F = \int_{BZ} \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{\sum_\mu i\gamma_\mu \frac{\sin(p_\mu a)}{a} + m + \frac{r}{a}\sum_\mu (1-\cos p_\mu a)}. \tag{4.48}$$

As before the integration over p_0 can be performed as a sum of residues of the solutions, in the region Im $p_0 > 0$, $-\pi <$ Re $p_0 < \pi$, of

$$\sum_\mu \sin^2 p_\mu + \left(m + \frac{r}{a}\sum_\mu (1-\cos p_\mu a)\right)^2 = 0. \tag{4.49}$$

For $r = 1$ (Wilson's choice) the only solution is at $p_0 = i\omega_{\mathbf{p}}$ satisfying

$$\cosh\omega_{\mathbf{p}} = \frac{1+\sum_k \sin^2 p_k a + (ma + 1 + \sum_k (1-\cos p_k a))^2}{2(ma+1+\sum_k (1-\cos p_k a)}. \tag{4.50}$$

The pole corresponding to the temporal doubler is absent. Also the spatial momenta of eq. (4.28), have an energy

$$\omega_{\mathbf{p}}^{(\alpha)} = \frac{1}{a}\ln\left(1 + ma + 2\sum_k n_k^{(\alpha)}\right), \tag{4.51}$$

therefore the only pole that survives in the continuum limit (i.e. $\lim_{a\to 0} a\omega_{\mathbf{p}} = 0$) corresponds to $n_k^{(\alpha)} = 0$ for all k. The others have energies of the order of the cutoff.

Wilson's solution therefore gets rid of doublers at the cost of breaking chiral symmetry.

Exercise 3.1 Symmetries of Wilson fermions. Show that the Wilson Dirac operator satisfies γ_5-hermiticity:

$$D^\dagger = \gamma_5 D \gamma_5$$

and is invariant under the discrete symmetries: C, P and T:

$$P : \psi(x) \to \gamma_0\psi(x_P) \tag{4.52}$$
$$\bar{\psi}(x) \to \bar{\psi}(x_P)\gamma_0 \tag{4.53}$$
$$T : \psi(x) \to \gamma_0\gamma_5\psi(x_T) \tag{4.54}$$
$$\bar{\psi}(x) \to \bar{\psi}(x_T)\gamma_5\gamma_0 \tag{4.55}$$
$$C : \psi(x) \to C\bar{\psi}^T(x) \tag{4.56}$$
$$\bar{\psi}(x) \to -\psi^T(x)C^{-1} \tag{4.57}$$

(4.58)

where $x_P = (x_0, -\mathbf{x})$, $x_T = (-x_0, \mathbf{x})$ and $C = \gamma_0\gamma_2$, which satisfies $C\gamma_\mu C = -\gamma_\mu^* = -\gamma_\mu^T$.

Exercise 3.2 Show that the γ_5-hermiticity implies that for the complex eigenvalues of D, the corresponding eigenvectors satisfy

$$v_\lambda^\dagger \gamma_5 v_\lambda = 0, \quad \lambda^* \neq \lambda. \tag{4.59}$$

Real eigenvalues on the other hand can have non-zero chirality.

4.3.1 Transfer matrix of Wilson fermions and unitarity

Actually, Wilson fermions with $r = 1$ are the only fermion regularization for which the transfer matrix has been proven to be positive (Wilson, 1975; Lüscher, 1977; Smit, 1991).

As in the scalar case, we will proceed by finding a transfer operator $\hat{T}$ acting on Fock space such that

$$\mathcal{Z}_F = \lim_{N\to\infty} \mathrm{Tr}[\hat{T}^N], \tag{4.60}$$

and proving that it is positive in such a way that the Hamiltonian $\hat{H} = -\frac{1}{a}\ln\hat{T}$ is well defined.

We need the equivalent to the Schrödinger representation of states. For the scalar field we defined the basis $|\phi\rangle$ (the analogue of the position basis in ordinary QM), such that

$$\hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle. \tag{4.61}$$

In the fermion case, similarly, we define a basis $|a\rangle$ (Smit, 2002), such that

$$\hat{a}_k|a\rangle = a_k|a\rangle, \tag{4.62}$$

where $\hat{a}_k$ are the annihilation operators in Fock space and a_k are Grassmann variables that represent the classical fermion field, which can be shown to anticommute with the operators.

One can show that the state $|a\rangle$ can be constructed from the vacuum as:

$$|a\rangle = \prod_k e^{-a_k \hat{a}_k^\dagger}|0\rangle. \tag{4.63}$$

Using the properties of the Grassmann integrals, one can also show that the basis $|a\rangle$ satisfies the completeness relation

$$\int da^\dagger da \frac{|a\rangle\langle a|}{\langle a|a\rangle} = 1, \tag{4.64}$$

where

$$\langle a|a\rangle = \prod_k e^{a_k^\dagger a_k} \equiv e^{a^\dagger a}, \quad a^\dagger a = \sum_k a_k^\dagger a_k. \tag{4.65}$$

Any arbitrary state in Fock space can be written as

$$|\psi\rangle = \sum_p \frac{1}{p!}\psi_{k_1,...k_p}\hat{a}^\dagger_{k_1}....\hat{a}^\dagger_{k_p}|0\rangle. \tag{4.66}$$

It has a wave function in the $|a\rangle$ basis:

$$\langle a|\psi\rangle \equiv \psi(a^\dagger) = \sum_p \frac{1}{p!}\psi_{k_1,...k_p}a^\dagger_{k_1}....a^\dagger_{k_p}. \tag{4.67}$$

Let us consider any normal-ordered operator $\hat{A}$

$$\hat{A} = \sum_{p,q}\frac{1}{p!q!}A_{k_1...k_p}\hat{a}^\dagger_{k_1}...\hat{a}^\dagger_{k_p}\hat{a}_{l_q}...\hat{a}_{l_1}. \tag{4.68}$$

The matrix elements of the operators in this basis can be shown to be:

$$A(a^\dagger, a) \equiv \langle a|\hat{A}|a\rangle = \langle a|a\rangle \sum_{p,q}\frac{1}{p!q!}A_{k_1...k_p}a^\dagger_{k_1}...a^\dagger_{k_p}a_{l_q}...a_{l_1}. \tag{4.69}$$

Finally, the following relations can also be derived (Smit, 2002):

- Trace:

$$\mathrm{Tr}\hat{A} \equiv \sum_p \frac{1}{p!}\sum_{k_1,...,k_p}\langle k_1,...,k_p|\hat{A}|k_1,...k_p\rangle = \int da^\dagger da e^{-a^\dagger a}A(a^\dagger, -a). \tag{4.70}$$

- The product of three operators, $\hat{A}, \hat{B}$ and $\hat{C}$, where $\hat{B}/\hat{C}$ only depend on creation/destruction operators respectively, while $\hat{A}$ depends on both, satisfies:

$$\langle a|\hat{B}\hat{A}\hat{C}|a\rangle = B(a^\dagger)A(a^\dagger, a)C(a). \tag{4.71}$$

- Operators of the exponential form

$$\hat{A} = \exp\left(\sum_{kl}\hat{a}^\dagger_k M_{kl}\hat{a}_l\right), \tag{4.72}$$

  satisfy

$$A(a^\dagger, a) = \exp\left(a^\dagger e^M a\right). \tag{4.73}$$

Let us see now how we can idenfity the transfer operator

$$\begin{aligned}\mathrm{Tr}[\hat{T}^N] &= \int da^\dagger_N da_N e^{-a^\dagger_N a_N}\langle a_N|\hat{T}^N| - a_N\rangle \\ &= \int \prod_n \left(da^\dagger_n da_n\right) e^{-a^\dagger_N a_N}\langle a_N|\hat{T}|a_{N-1}\rangle e^{-a^\dagger_{N-1}a_{N-1}}\langle a_{N-1}|\hat{T}|....|a_1\rangle e^{-a^\dagger_1 a_1}\langle a_1|\hat{T}| - a_N\rangle,\end{aligned} \tag{4.74}$$

that should be compared with eq. (4.18). As in the scalar case, we should somehow identify the a_n with the ψ at fixed times.

The Wilson fermion action (for $r = 1$) can be written as

$$S_W[\psi, \bar{\psi}] = a^3 \sum_{x_0} \sum_{\mathbf{x},\mathbf{y}} \psi^\dagger(\mathbf{x}, x_0)(\gamma_0 A_{\mathbf{xy}} + B_{\mathbf{xy}})\psi(\mathbf{y}, x_0)$$
$$+ a^4 \sum_{x_0,y_0} \sum_{\mathbf{x}} \psi^\dagger(\mathbf{x}, x_0) \frac{1}{2a} \left(P_- \delta_{y_0 x_0 + a} - P_+ \delta_{y_0 x_0 - a}\right) \psi(\mathbf{x}, y_0), \tag{4.75}$$

where

$$A_{\mathbf{xy}} \equiv (ma + 4)\,\delta_{\mathbf{xy}} - \frac{1}{2} \sum_k (\delta_{\mathbf{yx}+\hat{k}a} + \delta_{\mathbf{yx}-\hat{k}a}) \tag{4.76}$$

$$B_{\mathbf{xy}} \equiv \frac{1}{2} \sum_k \gamma_0 \gamma_k (\delta_{\mathbf{yx}+\hat{k}a} - \delta_{\mathbf{yx}-\hat{k}a}) \tag{4.77}$$

and $P_\pm = (1 \pm \gamma_0)/2$ are projectors in spinor space, with $P_+ + P_- = 1$.

Let us now decompose the fermions into their $\pm$ components and let us define a basis of the Grassmann variables a_{x_0} (we omit for simplicity the index that runs over $\mathbf{x}$ and the spinor indices) in the following way:

$$(a^\dagger_{x_0} P_+)^T \equiv P_+ \psi(x_0) a^{3/2}, \;\; P_- a_{x_0} \equiv P_- \psi(x_0 + a) a^{3/2}, \tag{4.78}$$
$$a^\dagger_{x_0} P_- \equiv \psi^\dagger(x_0) P_- a^{3/2}, \;\; (P_+ a_{x_0})^T \equiv \psi^\dagger(x_0 + a) P_+ a^{3/2}, \tag{4.79}$$

so that the $\pm$ components of ψ correspond to those of the a variables at different time slices. With these identifications, we can rewrite the action as

$$S_W[\psi, \bar{\psi}] = \sum_{x_0} \left(a^\dagger_{x_0} a_{x_0} - a^\dagger_{x_0} A a_{x_0 - a} + a_{x_0 - a} P_+ B P_- a_{x_0 - a} + a^\dagger_{x_0} P_- B P_+ a^\dagger_{x_0}\right) \tag{4.80}$$

Therefore, we find an exact matching if we identify $|a_n\rangle \to |a_{x_0}\rangle$ so that

$$e^{-\sum_{x_0} a^\dagger_{x_0} a_{x_0}} \to e^{-\sum_n a^\dagger_n a_n} \tag{4.81}$$

$$\langle a_n|\hat{T}|a_{n-1}\rangle \to \langle a_{x_0}|\hat{T}|a_{x_0-a}\rangle$$
$$= \exp(-a^\dagger_{x_0} P_+ B P_- a^\dagger_{x_0}) \exp(a^\dagger_{x_0} A a_{x_0 - a}) \exp(-a_{x_0 - a} P_+ B P_- a_{x_0 - a}), \tag{4.82}$$

which implies, according to eq. (4.73),

$$\hat{T} = \exp(-\hat{a}^\dagger P_+ B P_- \hat{a}^\dagger) \exp(\hat{a}^\dagger \ln(A) \hat{a}) \exp(-\hat{a} P_- B P_+ \hat{a}). \tag{4.83}$$

Probing the positivity is now straightforward. The operator in the middle is positive if A is positive. In momentum space the operator is

$$A(p) = ma + 4 + \sum_k \cos p_k a > 0. \tag{4.84}$$

Since the transfer matrix has the structure

$$\hat{T} = \hat{T}_1^\dagger \hat{T}_2 \hat{T}_1, \tag{4.85}$$

for any state $|\psi\rangle$

$$\langle\psi|\hat{T}_1^\dagger \hat{T}_2 \hat{T}_1|\psi\rangle = \langle\xi|\hat{T}_2|\xi\rangle > 0, \quad |\xi\rangle = \hat{T}_1|\psi\rangle, \tag{4.86}$$

and $\hat{T}$ is a positive Hermitian operator, from which a Hermitian Hamiltonian can be defined. Therefore the lattice formulation of Wilson fermions with $r = 1$ has a direct Hilbert space interpretation, just as the scalar theory.

The case $r \neq 1$ cannot be treated in the same way, and in fact positivity has not been proven. Reflection positivity on the other hand can be proved for $r \leq 1$ (Osterwalder and Seiler, 1978; Menotti and Pelissetto, 1987).

4.4 Kogut-Susskind or staggered fermions

Given that naive lattice fermions correspond to 2^d Dirac fermions in the continuum, one idea would be to use some of the doublers to represent the $4 = 2^{d/2}$ spinor components of a Dirac fermion. Kogut and Susskind (Kogut and Susskind, 1975) proved that this can be done, therefore reducing the doubling problem to that of $2^d/2^{d/2}$ replicas instead of 2^d. The advantage is that the lattice action can be shown to have an extra exact $U(1)$ symmetry compared to the Wilson action.

Let us briefly review the construction of the Kogut-Susskind action. There are two steps

- Perform a local unitary rotation of the fermion fields that diagonalizes the action in spinor space. That is, find a unitary S_x such that

$$S_x^\dagger \gamma_\mu S_{x+\hat{\mu}a} = \rho_{x\mu} \mathbf{I}. \tag{4.87}$$

It is easy to prove that the choice

$$S_x \equiv \gamma_0^{n_0} ... \gamma_3^{n_3} = \prod_\mu \gamma_\mu^{n_\mu} \quad x = a\ (n_0, n_1, n_2, n_3) \tag{4.88}$$

satisfies eq.(4.87) with $\rho_{x\mu} = (-1)^{\sum_{\rho<\mu} n_\rho}$. We can therefore perform the transformation of the spinors

$$\psi(x)_\alpha \to (S_x)_{\alpha\beta} \psi_\beta(x) \equiv \chi_\alpha(x), \tag{4.89}$$

and the action factorizes in the four spinor components

$$S_{KS} = a^4 \sum_{x,\alpha} \left[\sum_\mu \rho_{x\mu} \bar{\chi}^\alpha(x) \frac{1}{2} \left(\chi^\alpha(x + a\hat{\mu}) - \chi^\alpha(x - a\hat{\mu})\right) + m\bar{\chi}^\alpha(x)\chi^\alpha(x) \right] \tag{4.90}$$

We can therefore consider just *one* of this replicas that we call χ.

- Reconstruction of the Dirac field

In order to reconstruct the Dirac field using the doublers associated to the variable χ, one needs to consider a lattice with a doubled lattice spacing $2a$. The Dirac fields will be Grassmann variables defined on this coarser lattice. We can define therefore coordinates in the new lattice as

$$y_\mu = 2aN_\mu, \tag{4.91}$$

while the coordinates of the points in the original lattice can be labelled as

$$x = an_\mu = 2aN_\mu + az_\mu, \quad z_\mu = 0, 1. \tag{4.92}$$

Therefore we define new fields on the coarser lattice that will have a space-time coordinates $y_\mu = 2aN_\mu$ and also other 2^d internal components labelled by z_μ:

$$\chi(na) \equiv \psi_z(2Na). \tag{4.93}$$

It is easy to show that

$$\chi(n+\hat{\mu}) = \sum_{z'} \delta_{z+\hat{\mu}z'}\psi_{z'}(N) + \delta_{z-\hat{\mu}z'}\psi(N+\hat{\mu})_{z'} \tag{4.94}$$

$$\chi(n-\hat{\mu}) = \sum_{z'} \delta_{z-\hat{\mu}z'}\psi_{z'}(N) + \delta_{z+\hat{\mu}z'}\psi(N-\hat{\mu})_{z'}. \tag{4.95}$$

Defining

$$\Gamma^{\mu}_{zz'} \equiv \rho_{z\mu}(\delta_{z+\hat{\mu}z'} + \delta_{z-\hat{\mu}z'})\Gamma^{5\mu}_{zz'} \equiv \rho_{z\mu}(\delta_{z-\hat{\mu}z'} - \delta_{z+\hat{\mu}z'}) \tag{4.96}$$

where

$$\rho_{z\mu} = (-1)^{\sum_{\nu\leq\mu} z_\nu} \tag{4.97}$$

it is easy to show that in terms of the new fields the action is

$$S_{KS} = a^4 \sum_{N,z}\sum_{\mu} \bar{\psi}_z(N)\frac{1}{4}\left[\Gamma^{\mu}_{zz'}(\hat{\partial}_\mu + \hat{\partial}^*_\mu) + \Gamma^{5\mu}_{zz'}a\hat{\partial}^*_\mu\hat{\partial}_\mu\right]\psi_{z'}(N), \tag{4.98}$$

where $\hat{\partial}_\mu, \hat{\partial}^*_\mu$ are the forward and backward derivatives in the coarser lattice.

Furthermore, we can show that

$$\Gamma^{\mu}_{zz'} = \mathrm{Tr}\left[S^\dagger_z\gamma_\mu S_{z'}\right], \tag{4.99}$$

$$\Gamma^{5\mu}_{zz'} = \mathrm{Tr}\left[S^\dagger_z\gamma_5 S_{z'}\gamma_5\gamma_\mu\right]. \tag{4.100}$$

with

$$S_z \equiv \prod_\nu \gamma_\nu^{z_\nu}. \tag{4.101}$$

Finally, defining

$$\Psi^{\alpha i}(N) \equiv \sum_z (S_z)_{\alpha i}\psi_z(N) \quad \bar{\Psi}^{\alpha i}(N) \equiv \sum_z \bar{\psi}_z(N)\left(S^\dagger_z\right)_{i\alpha}, \tag{4.102}$$

we can get back four Dirac spinors with spinor index α and flavour index i. After a simple normalization we get the Kogut-Susskind action in terms of the new variables:

$$S_{KS} = (2a)^4 \sum_{\mu,N}\left[\bar{\Psi}(N)(\gamma_\mu\otimes 1)\frac{1}{2}(\hat{\partial}_\mu + \hat{\partial}^*_\mu)\Psi(N) + a\bar{\Psi}(N)(\gamma_5\otimes\gamma^T_\mu\gamma^T_5)\frac{1}{2}a\ \hat{\partial}_\mu\hat{\partial}^*_\mu\Psi(N)\right]$$

$$+ (2a)^4 m \sum_N \bar{\Psi}(N)\Psi(N), \tag{4.103}$$

where the first γ matrices act on spinor variables, and the second in flavour ones.

A few comments are in order:

- In the naive continuum limit, the "Wilson"-type term vanishes and the action goes to the continuum action of four free massive Dirac spinors.
- The action looks quite similar to the Wilson action. The difference is the Dirac/flavour structure of the Wilson term.
- The action has an exact $U(1)$ chiral symmetry for $m = 0$ under spin-flavour rotations of the form

$$\Psi_N \to e^{i\alpha(\gamma_5 \otimes \gamma_5^T)}\Psi_N, \quad \bar{\Psi}_N \to \bar{\Psi}_N e^{i\alpha(\gamma_5 \otimes \gamma_5^T)}. \tag{4.104}$$

  This symmetry can be preserved in the interacting case, and ensures a chiral symmetry in the continuum limit without extra fine-tunings.

The transfer matrix operator has been constructed also for staggered fermions (Smit, 2002), but it is not positive, therefore there is no warranty that the formulation has a Hilbert space formulation at finite lattice spacing. The hope is therefore that unitarity is recovered in the continuum limit, which seems to be the case in the lowest orders of perturbation theory. Further subtleties of staggered fermions in the interacting case can be found in the lectures of M. Golterman (Golterman, 2009).

Recently very significant progress has been achieved in the constructions of fermion actions that preserve a lattice chiral symmetry. These new developments are covered in the lectures of D. Kaplan (Kaplan, 2009).

Exercise 3.3 Two flavours of twisted-mass Wilson fermions are defined by the Wilson action with a mass term that has the form

$$m + i\mu\gamma_5\tau_3, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.105}$$

Show that in the naive continuum limit the action is equivalent to the standard Dirac action by performing a chiral rotation of the form

$$\psi \to e^{i\frac{\alpha}{2}\gamma_5\tau_3}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\frac{\alpha}{2}\gamma_5\tau_3}\psi, \quad \tan\alpha = \frac{\mu}{m}. \tag{4.106}$$

5
Lattice Gauge Fields

5.1 Lattice gauge field theories: abelian case

K. Wilson figured out how to formulate a quantum field theory of gauge fields on the lattice preserving an exact gauge invariance (Wilson, 1974).

We will first derive the lattice formulation of an abelian gauge theory and then we will generalize the construction to other gauge theories such as $SU(3)$ describing the color interactions.

An easy way to understand how this is done is to consider the case of charged particles (scalar ones to make it simple). A scalar charged particle is described by a complex scalar field. The results of chapter 3 can be readily applied to a complex scalar field.

It is well known that gauge invariance in quantum mechanics corresponds to a symmetry under local rephasing of the wave functions describing the charged particles, and a shift of the gauge potentials. Maxwell equations are invariant under

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x), \tag{5.1}$$

while the Schrödinger equation describing a particle with charge q in this field is also invariant if there is a simultaneous rephasing of the charged field wave function by

$$\phi(x) \to e^{iq\Lambda(x)}\phi(x) \equiv \Omega(x)\phi(x). \tag{5.2}$$

Let us consider the case in which the electromagnetic field strength vanishes in all space, that is we consider a *pure gauge* configuration, i.e. $A_\mu = \partial_\mu F(x)$ for arbitrary $F(x)$. We can then choose a gauge in which the gauge potential vanishes. In this gauge it is easy to discretize the scalar action, it is just the one corresponding to a free scalar field:

$$S = \frac{a^4}{2}\sum_{x,y} \phi^\dagger(x) K_{xy} \phi(y), \tag{5.3}$$

with

$$K_{xy} = -\frac{1}{a^2}\sum_{\hat\mu}\left(\delta_{x+a\hat\mu y} + \delta_{x-a\hat\mu y} - 2\delta_{xy}\right) + m^2 \delta_{xy}. \tag{5.4}$$

Now, let us change the gauge, which implies a rephasing of the charged fields,

$$\phi(x) \to e^{iq\Lambda(x)}\phi(x) = \phi'(x) \quad \phi(x)^\dagger \to \phi(x)^\dagger e^{-iq\Lambda(x)} = \phi'(x)^\dagger, \tag{5.5}$$

and a change of the gauge field to $A'_\mu = \partial_\mu \Lambda(x)$.

The action in terms of the new fields, $\phi'(x)$, should therefore correspond to the action of a scalar field coupled to the gauge field $A'_\mu = \partial_\mu \Lambda$. Substituting $\phi(x)$ in terms of $\phi'(x)$ in eq. (5.3), we find:

$$S = \frac{a^4}{2} \sum_{x,y} \phi'^\dagger(x) K^\Lambda_{xy} \phi'(y), \tag{5.6}$$

where

$$K^\Lambda_{xy} = -\frac{1}{a^2} \sum_{\hat{\mu}} \left(\delta_{x+a\hat{\mu}y} U_\mu(x) + \delta_{x-a\hat{\mu}y} U^\dagger_\mu(x - a\hat{\mu}) - 2\delta_{xy} \right) + m^2 \delta_{xy}. \tag{5.7}$$

We have introduced the so-called *link variables*, defined as

$$U_\mu(x) \equiv e^{iq\Lambda(x)} e^{-iq\Lambda(x+a\hat{\mu})} = e^{-iq \int_x^{x+a\hat{\mu}} \partial_\mu \Lambda(x) dx_\mu} \equiv e^{-iq \int_x^{x+a\hat{\mu}} dx_\mu A'_\mu(x)}. \tag{5.8}$$

The link variable is nothing but a *parallel transporter* between two adjacent points on the lattice, $x + a\hat{\mu}$ and x:

$$P(x, a + a\hat{\mu}) \equiv \exp\left(iq \int_{x+a\hat{\mu}}^{x} A'_\mu(x) dx_\nu \right). \tag{5.9}$$

The integral can be done along the straight line:

$$x_\mu(t) = x + ta\hat{\mu}, \quad t = [0, 1], \tag{5.10}$$

and this is why we associate it to a link. In the following, we will absorb the charge q in the gauge potential.

We can now check that the action in eq. (5.6) in terms of A'_μ is gauge invariant for any gauge field (not just the pure gauge configurations we started with). Consider a general continuum gauge field $A_\mu(x)$, not necessarily with vanishing field strength. The gauge transformation of a parallel transporter between points x and y is

$$\begin{aligned} P^\Lambda(y, x) &= \exp\left(i \int_x^y (A_\mu + \partial_\mu \Lambda) dx_\mu \right) = \exp\left(i \int_x^y A_\mu dx_\mu + i\Lambda(y) - i\Lambda(x) \right) \\ &= \exp(i\Lambda(y)) P(y, x) \exp(-i\Lambda(x)) = \Omega(y) P(y, x) \Omega^\dagger(x). \end{aligned} \tag{5.11}$$

The lattice action of eq. (5.6) is indeed invariant under the gauge transformation

$$\phi'(x) \to \Omega(x) \phi'(x) \quad U_\mu(x) \to \Omega(x) U_\mu(x) \Omega^\dagger(x + a\hat{\mu}). \tag{5.12}$$

It is easy to generalize this procedure to fermions or any other charged fields. Starting with the free action we can couple the field to a gauge field by substituting the partial derivatives by covariant ones:

$$\hat{\partial}_\mu \psi(x) = \frac{1}{a} \left(\psi(x + a\hat{\mu}) - \psi(x) \right) \to \nabla_\mu \psi(x) = \frac{1}{a} \left(U_\mu(x) \psi(x + a\hat{\mu}) - \psi(x) \right),$$

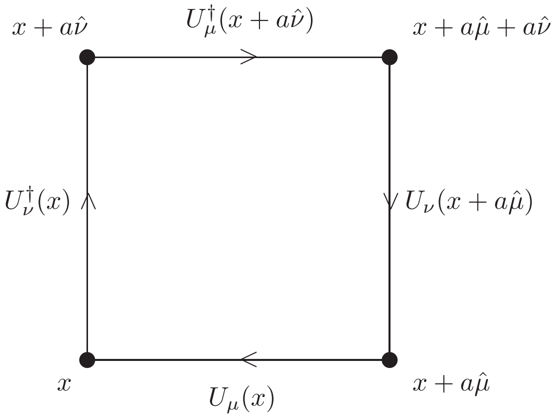


Fig. 5.1 Plaquette

$$\hat{\partial}_\mu^* \psi(x) = \frac{1}{a}\left(\psi(x) - \psi(x - a\hat{\mu})\right) \to \nabla_\mu^* \psi(x) = \frac{1}{a}\left(\psi(x) - U_\mu^\dagger(x - a\hat{\mu})\psi(x - a\hat{\mu})\right). \tag{5.13}$$

5.1.1 Path integral

Now that we have identified the parallel transporters as the basic gauge variables on the lattice, we still need to construct the Euclidean lattice path integral to represent the continuum one

$$\mathcal{Z} = \int dA_\mu e^{-S[A_\mu]} \quad S[A_\mu] \equiv \frac{1}{4}\int d^4x F_{\mu\nu}F_{\mu\nu}, \tag{5.14}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. Obviously we must do this ensuring that eq. (5.12) remains a symmetry.

Let us consider any ordered loop of parallel transporters, a so-called *Wilson loop*. A Wilson loop starting and ending in the point x transforms as

$$W(x) \equiv P(x, y_1)P(y_1, y_2)....P(y_n, x) \to \Omega(x)W(x)\Omega(x)^\dagger. \tag{5.15}$$

In the case of an abelian group, $W(x)$ is therefore invariant.

Since we want our action to be local, we can try with the smallest Wilson loop, which is a loop around the basic plaquette, see Fig. 5.1:

$$U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\mu^\dagger(x), \tag{5.16}$$

indeed under a gauge transformation

$$U_{\mu\nu}(x) \to \Omega(x)U_{\mu\nu}(x)\Omega(x)^\dagger. \tag{5.17}$$

It is an easy exercise to check that if we define a lattice gauge field $\hat{A}_\mu$ by

$$U_\mu \equiv e^{iqa\hat{A}_\mu(x)}, \tag{5.18}$$

then

$$U_{\mu\nu}(x) = e^{-iqa^2\hat{F}_{\mu\nu}+\mathcal{O}(a^3)}, \tag{5.19}$$

where

$$\hat{F}_{\mu\nu}(x) \equiv \hat{\partial}_\mu \hat{A}_\nu(x) - \hat{\partial}_\nu \hat{A}_\mu(x), \tag{5.20}$$

and the derivatives are discrete ones, eq. (3.8).

From this result it is easy to guess a good lattice action for the link variables:

$$S[U] = \frac{1}{q^2}\sum_x \sum_{\mu\leq\nu}\left[1 - \frac{1}{2}\left(U_{\mu\nu}(x) + U^\dagger_{\mu\nu}(x)\right)\right], \tag{5.21}$$

which satisfies the basic properties:

- It is local
- It is real
- It is gauge invariant
- It has the right classical continuum limit [1]:

$$\lim_{a\to 0} S[U] = \int d^4x \frac{1}{4} F^2_{\mu\nu} + \mathcal{O}(a^2) \tag{5.22}$$

We still need to define the measure over the link variables. Since the link variables are elements of $U(1)$ we can define a gauge invariant measure as

$$dU \equiv \prod_{\mu,x} d\phi_\mu(x) \quad U_\mu(x) = e^{i\phi_\mu(x)}, \quad 0 \leq \phi_\mu(x) \leq 2\pi. \tag{5.23}$$

Since the variables at different points are independent, and a gauge transformation induces a constant shift of the phase of each link variable,

$$\phi_\mu(x) \to \phi'_\mu(x) = \Lambda(x) + \phi_\mu(x) - \Lambda(x + a\hat{\mu}), \quad d\phi_\mu(x) = d\phi'_\mu(x), \tag{5.24}$$

this measure is gauge invariant. We will see that the measure is less trivial in the non-abelian case.

5.2 Lattice gauge field theories: non-abelian case

The colour interactions in QCD are based on the non-abelian gauge symmetry $SU(3)$. In the continuum, the Yang-Mills theory based on a group $SU(N)$ is a quantum field

[1] Whether a continuum limit of this discretized theory exists is of course not warrantied from this property.

theory of the vector gauge potential $A_\mu(x)$ that takes values in the Lie algebra of the gauge group:

$$A_\mu(x) = A_\mu^a(x)T^a, \tag{5.25}$$

where the coefficients $A_\mu^a(x)$ are real and $T^a = (T^a)^\dagger$ are the Hermitian generators of the algebra. The Yang-Mills field tensor is defined by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i[A_\mu(x), A_\nu(x)], \tag{5.26}$$

which is also an element of the algebra.

A gauge transformation is:

$$A_\mu(x) \to \Omega(x)A_\mu(x)\Omega(x)^{-1} + i\Omega(x)\partial_\mu\Omega(x)^{-1}, \tag{5.27}$$

where $\Omega(x) \in SU(N)$. It implies the following transformation of the field tensor

$$F_{\mu\nu}(x) \to \Omega(x)F_{\mu\nu}(x)\Omega(x)^{-1}. \tag{5.28}$$

The Euclidean Yang-Mills action is given by

$$S[A_\mu] = \frac{1}{2g_0^2}\int d^4x \mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right], \tag{5.29}$$

and is therefore gauge invariant.

A colored scalar field in the fundamental representation of this symmetry group transforms as

$$\phi(x) \to \phi'(x) = \Omega(x)\phi(x) \quad \Omega \in SU(N). \tag{5.30}$$

The main difference with the $U(1)$ case is that now the Ω are $N \times N$ matrices that do not commute.

We can proceed as for the $U(1)$ case above and start with the free action of colored scalar fields eq. (5.3). We can then identify the way gauge fields appear in the lattice action by performing a gauge transformation of the coloured fields, $\phi(x) \to \phi'(x)$. The field $\phi'(x)$ will then be coupled to a gauge field, according to eq. (5.27),

$$A'_\mu(x) = i\Omega(x)\partial_\mu\Omega(x)^{-1}. \tag{5.31}$$

When we do this, we find the same result as in the $U(1)$ case, eq. (5.6), provided we define the link variables as

$$U_\mu(x) \equiv \Omega(x)\Omega(x + a\hat{\mu})^\dagger. \tag{5.32}$$

This corresponds to the parallel transporter of the non-abelian gauge field eq. (5.31) from $x + a\hat{\mu}$ to x.

To see this, let us recall the definition of a parallel transporter for $SU(N)$. Consider a N component vector $\mathbf{v}$ of unit length and a curve in R^4 that can be parametrized by

$z_\mu(t)$. A parallel transport of $\mathbf{v}$, along such a curve from points t_0 to t, in the presence of the field A_μ is the solution of the equation

$$\left[\frac{d}{dt} - i\frac{dz_\mu(t)}{dt}A_\mu(z_\mu(t))\right]\mathbf{v}(t) = 0. \tag{5.33}$$

The parallel transporter from $z_\mu(t_0) = x$ to $z_\mu(t) = y$, $P(y,x)$, is the matrix that satisfies

$$\mathbf{v}(t) = P(y,x)\mathbf{v}(t_0). \tag{5.34}$$

In the abelian case, the solution to this equation is eq. (5.9). For the non-abelian case, the solution can be written as a series in A_μ:

$$\begin{aligned}\mathbf{v}(t) = &\left(I + i\int_0^t dt_1 \dot{z}_\mu(t_1)A_\mu(z(t_1))\right.\\ &\left.- \int_0^t dt_1 \dot{z}_\mu(t_1)A_\mu(z(t_1))\int_0^{t_1} dt_2 \dot{z}_\nu(t_2)A_\nu(z(t_2)) + ...\right)\mathbf{v}(t_0)\\ \equiv\ &P\exp\left(i\int_x^y A_\mu(z)dz_\mu\right)\mathbf{v}(t_0).\end{aligned} \tag{5.35}$$

Now it is easy to check (see exercise), using the definition, eqs. (5.33) and (5.34), that the parallel transporter from two adjacent points on the lattice $x + a\hat{\mu}$ and x in the vector potential of eq. (5.31) is given by eq. (5.32).

Similarly from the definition it is easy to show that the gauge transformation of a parallel transporter is

$$P(y,x) \to \Omega(y)P(y,x)\Omega^\dagger(x). \tag{5.36}$$

Exercise 4.1 Prove, using the definition of the parallel transporter of eq. (5.33), that

- the link variable

$$U_\mu(x) \equiv \Omega(x)\Omega(x + a\hat{\mu})^\dagger, \tag{5.37}$$

  is a parallel transporter from $x + a\hat{\mu}$ to x
- the gauge transformation of a parallel transporter is

$$U(x,y) \to \Omega(x)U(x,y)\Omega^\dagger(y). \tag{5.38}$$

These properties are sufficient to ensure the gauge invariance of the plaquette action also for $SU(N)$:

$$S[U] \equiv C \sum_x \sum_{\mu<\nu} \mathrm{Tr}\left[1 - \frac{1}{2}\left(U_{\mu\nu}(x) + U^\dagger_{\mu\nu}(x)\right)\right]. \qquad (5.39)$$

The coefficient C can be chosen to recover the conventional normalization in the classical continuum limit, eq. (5.29):

$$C \equiv \frac{2}{g_0^2}, \qquad (5.40)$$

where g_0 is the gauge coupling.

5.2.1 Gauge Measure and Path integral

In order to define the path integral we still need to define the measure over the link variables in a gauge invariant way. Since the link variables are elements of a compact group $SU(N)$, the measure is nothing but the Haar measure on the group, which can be proven to be the unique measure which obeys two essential properties

- it is gauge invariant

$$\int_{SU(N)} dU f(U) = \int_{SU(N)} f(VU)dU = \int_{SU(N)} f(UV)dU, \qquad (5.41)$$

  for any $V \in SU(N)$
- it is normalized

$$\int_{SU(N)} dU = 1. \qquad (5.42)$$

Let us consider any parametrization of the group in terms of n coordinates, z_i, then

$$dU = w(z)dz_1dz_2...dz_n, \qquad (5.43)$$

where n must be the number of generators of the algebra. The invariance of the measure requires

$$dU(z') = d(VU(z)W^\dagger) = dU(z), \qquad (5.44)$$

for arbitrary $V, W \in SU(N)$. Therefore

$$w(z')dz'_1...dz'_n = w(z')|\det(\partial z'_a/\partial z_b)|dz_1....dz_n = w(z)dz_1...dz_n \qquad (5.45)$$

or

$$w(z') = w(z)/J(z,z'), \qquad (5.46)$$

where $J(z,z')$ is the Jacobian of the transformation $z \to z'$. If there were two different measures satisfying this property, the only possibility is that both functions are proportional up to a constant. The constant is then fixed by the normalization condition and the measure is unique given a set of coordinates.

Once we are sure that the measure is unique, we can find it by explicit construction. We can define a metric tensor in the group by

$$g_{kl} \equiv -2\text{Tr}[(U\partial_k U^{-1})(U\partial_l U^{-1})], \tag{5.47}$$

which can be shown to be positive definite and gauge invariant. The measure in this coordinates can then be defined as

$$w(z) = c\sqrt{\det g(z)}, \tag{5.48}$$

where c is obtained from the normalization condition.

Exercise 4.2 Using the Haar measure, eqs. (5.47) and (5.48), show that

$$\int dU f(U) = \int dU f(U^*) = \int dU f(U^{-1}). \tag{5.49}$$

We will now show a few of the most commonly used coordinates in the simplest case of $SU(2)$.

5.2.2 Examples of coordinate systems for $SU(2)$

1) For $SU(2)$ a useful parametrization maps the group elements to a three-dimensional sphere S^3:

$$U = x_0 + ix_a\sigma_a, \qquad x^2 = x_0^2 + \sum_{a=1}^{3} x_a^2 = 1, \tag{5.50}$$

where σ_a are the Pauli matrices. The Haar measure is simply

$$dU = \frac{1}{\pi^2}\delta(x^2 - 1)d^4x \tag{5.51}$$

2) From a sphere S^3 we can easily go to R^3 via a stereographic projection, leaving undefined only the element at the north pole $U = -1$. The stereographic coordinates, $\mathbf{z} = (z_1, z_2, z_3)$, can be related to those on the sphere by

$$x_0 = \frac{(1-\mathbf{z}^2)}{(1+\mathbf{z}^2)}, \qquad x_a = \frac{2z_a}{(1+\mathbf{z}^2)}. \tag{5.52}$$

The Haar measure is

$$dU = d^3z\frac{4}{\pi^2(1+\mathbf{z}^2)^3}. \tag{5.53}$$

3) Finally the exponential mapping that is useful in perturbation theory

$$U = \exp(i\phi n_a \sigma^a/2) = \cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\mathbf{n}\cdot\sigma, 0 \leq \phi \leq 2\pi, \tag{5.54}$$

where $\mathbf{n}$ are unit vectors in R^3. The invariant measure is

$$dU = \frac{1}{4\pi^2} d\phi\, d\Omega(\mathbf{n}) \sin(\phi/2)^2, \tag{5.55}$$

where $\Omega(\mathbf{n})$ is the uniform measure in S^2.

Defining a global coordinate system for $SU(N)$ is more complicated. Very often however explicit expressions of the measure are not needed, because integrals can be solved by invariant tensor methods.

Exercise 4.3 Work out the Haar measure in $SU(2)$ in terms of the variables α_k:

$$U = \exp(i\alpha_k \sigma_k), \tag{5.56}$$

where σ_k are the Pauli matrices. Compute the constant c so that the measure is properly normalized.

Two observations are in order:

- the integrals over the link variables are finite, there is no need to fix the gauge
- the integrals can be done via importance sampling methods, because the action is real and positive definite

$$S[U] \sim \sum_P \mathrm{Tr}[2 - U_P - U_P^\dagger] = \sum_P \mathrm{Tr}[(1 - U_P)(1 - U_P^\dagger)] \geq 0, \tag{5.57}$$

the equality being obtained only when all plaquettes are unity: $U_P = 1$.

Before including the sources we are interested in, we should find out what are the operators that should represent the particle excitations in this theory. In order to understand this we should make contact with the operator formulation via the transfer matrix, which defines the Hamiltonian.

5.3 Transfer matrix and unitarity of the plaquette action

As for the scalar and fermion lattice field theories, we want to make sure that there is a Hilbert space representation of the lattice gauge theory. We need therefore to identify the field operators at $t = 0$ that represent creation and annihilation of particles. We also need to identify the transfer operator that evolves the operators at $t = 0$ in time.

In the Schrödinger picture, the physical states are described by wave functions that depend on the basic field variables at time $t = 0$. The time evolution of these fields is related to the Hamiltonian. At a fixed time $t = 0$, we can identify the spatial links:

$$U_k(\mathbf{x}, 0) \quad k = 1, 2, 3 \tag{5.58}$$

with the wave function coordinates, so the Schröndiger wave function of an arbitrary state $|\psi\rangle$ in this basis depends only on these links:

$$\psi[U_k(\mathbf{x}, 0)] = \langle U|\psi\rangle, \tag{5.59}$$

where $|U\rangle$ are the eigenbasis of the spatial link operators (i.e. analogous to the position basis in ordinary QM):

$$\hat{U}_k(\mathbf{x})|U\rangle = U_k(\mathbf{x}, 0)|U\rangle. \tag{5.60}$$

The states $|U\rangle$ form an orthonormal and complete basis

$$\langle U|U'\rangle = \prod_{\mathbf{x},k} \delta(U'_k(\mathbf{x}, 0) - U_k(\mathbf{x}, 0)) \tag{5.61}$$

with

$$\int dU \delta(U, U') = 1. \tag{5.62}$$

The scalar product of two such wave functions is therefore

$$\langle\psi|\phi\rangle \equiv \int \prod_{\mathbf{x},k} dU_k(\mathbf{x}) \; \psi[U]^\dagger \phi[U]. \tag{5.63}$$

A gauge transformation leaves the scalar product invariant thanks to the invariance of the Haar measure and therefore the symmetry transformation must correspond to a unitarity operator,

$$\psi[U^\Omega] = \omega\psi[U], \tag{5.64}$$

where ω is the unitary operator that implements the gauge transformation in the space of wave functions.

In contrast with the ϕ^4 model previously discussed however, the Hilbert space of physical states includes only those wave functions that are gauge invariant:

$$\psi[U^\Omega] = \psi[U]. \tag{5.65}$$

We can define a projector on gauge-invariant wave functions in the following way

$$\psi_{phys}[U] = \mathcal{P}_{phys}\psi[U] = \int \prod_{\mathbf{x}} d\Omega(\mathbf{x})\psi[U^\Omega]. \tag{5.66}$$

It is trivial to check, using the invariance of the measure, that the wave function $\psi_{phys}[U]$ is gauge invariant. Also that the projector acts trivially on gauge-invariant wave functions.

The transfer matrix, $\hat{T}$, is an operator in the Hilbert space of wave functions that must satisfy that

$$\mathcal{Z} = \lim_{N\to\infty} \mathrm{Tr}[\hat{T}^N], \qquad N = T/a. \tag{5.67}$$

We can rewrite, inserting the completeness relation for the $|U\rangle$, basis:

$$\int dU^{(m)} |U^{(m)}\rangle\langle U^{(m)}| = 1, \tag{5.68}$$

$$\begin{aligned}\mathrm{Tr}[\hat{T}^N] &= \int dU^{(0)} \int dU^{(1)} ... \langle U^{(0)}|\hat{T}|U^{(1)}\rangle\langle U^{(1)}|\hat{T}|.....|U^{(N-1)}|\hat{T}|U^{(0)}\rangle \\ &= \prod_{m=0}^{N-1} \int dU^{(m)} \langle U^{(m)}|\hat{T}|U^{(m+1)}\rangle,\end{aligned} \tag{5.69}$$

where $|U^{(m)}\rangle$ are basis states at the time slice $x_0 = mT/N = ma$ and $|U^{(N)}\rangle = |U^{(0)}\rangle$. We can therefore identify the coordinates $U_k^{(m)}(\mathbf{x})$ with the spatial links at time ma:

$$U_k^{(m)}(\mathbf{x}) \to U_k(\mathbf{x}, ma), \tag{5.70}$$

with periodic boundary conditions in time.

Let us rewrite the plaquette action in the following way

$$\begin{aligned}S[U] = \frac{1}{g_0^2} \sum_m \sum_{\mathbf{x}} &\left(\sum_{k<l} \mathrm{Tr}[2 - U_{kl}(\mathbf{x}, ma) - U_{kl}^\dagger(\mathbf{x}, ma)] \right. \\ &+ \left. \sum_k \mathrm{Tr}[2 - U_{k0}(\mathbf{x}, ma) - U_{k0}^\dagger(\mathbf{x}, ma)] \right) \\ &= \sum_m \left\{ V[U^{(m)}] + K[U^{(m)}, U^{(m+1)}] + V[U^{(m+1)}] \right\},\end{aligned} \tag{5.71}$$

where

$$V[U^{(m)}] \equiv \frac{1}{2g_0^2} \sum_{\mathbf{x}} \sum_{k<l} \mathrm{Tr}[2 - U_{kl}(\mathbf{x}, ma) - U_{kl}^\dagger(\mathbf{x}, ma)], \tag{5.72}$$

$$K[U^{(m)}, U^{(m+1)}] \equiv \frac{1}{g_0^2} \mathrm{Tr}[2 - (U_k(\mathbf{x}, ma)U_0(\mathbf{x} + a\hat{k}, ma)U_k^\dagger(\mathbf{x}, ma + a)U_0^\dagger(\mathbf{x}, ma) + h.c.)]. \tag{5.73}$$

Now we can rewrite the path integral separating the integration over spatial and temporal links:

$$\mathcal{Z} = \prod_m \int \prod_{\mathbf{x},k} dU_k(\mathbf{x}, ma) dU_0(\mathbf{x}, ma) \exp\left[-(V[U^m] + K[U^{(m)}, U^{(m+1)}] + V[U^{(m+1)}]) \right],$$

(5.74)

which can be written in the form of eq. (5.69) if we identify

$$\langle U^{(m)}|\hat{T}|U^{(m+1)}\rangle = \int \prod_{\mathbf{x}} dU_0(\mathbf{x}, ma) \exp\left[-(V[U^m] + K[U^{(m)}, U^{(m+1)}] + V[U^{m+1}])\right]. \tag{5.75}$$

Therefore the operator $\hat{T}$ has the form

$$\hat{T} = e^{-\hat{V}}\hat{T}_K e^{-\hat{V}}, \tag{5.76}$$

where $\hat{V}$ is an Hermitian operator diagonal in the $|U\rangle$ basis

$$\langle U'|\hat{V}|U\rangle = V[U]\delta(U', U), \tag{5.77}$$

and $V[U]$ is defined in eq. (5.72). The kinetic operator satisfies

$$\langle U'|\hat{T}_K|U\rangle = \int \prod_{k,\mathbf{x}} d\Omega(\mathbf{x}) \exp\left(-\frac{1}{g_0^2}\mathrm{Tr}[2 - (U'_k(\mathbf{x})\Omega(\mathbf{x}+a\hat{k})U_k^\dagger(\mathbf{x})\Omega^\dagger(\mathbf{x}) + h.c.)]\right). \tag{5.78}$$

We define the operator $\hat{T}_K^0$:

$$\langle U'|\hat{T}_K^0|U\rangle \equiv \exp\left(-\frac{1}{g_0^2}\mathrm{Tr}[2 - (U'_k(\mathbf{x})U_k^\dagger(\mathbf{x}) + h.c.)]\right). \tag{5.79}$$

It is easy to show that

$$\langle U'|\hat{T}_K|U\rangle = \langle U'|\hat{T}_K^0\mathcal{P}_{phys}|U\rangle = \langle U'|\mathcal{P}_{phys}\hat{T}_K^0|U\rangle, \tag{5.80}$$

for all $|U\rangle, |U'\rangle$, where $\mathcal{P}_{phys}$ is the projector we have defined in eq. (5.66). From this we can easily show two important properties:

- $\hat{T}_K$ commutes with the projector on physical states, and therefore only transforms physical states (gauge invariant under time independent gauge transformations) to physical states:

$$\hat{T}_K\mathcal{P}_{phys} = \hat{T}_K^0\mathcal{P}_{phys}^2 = \hat{T}_K^0\mathcal{P}_{phys} = \hat{T}_K = \mathcal{P}_{phys}\hat{T}_K^0 = \mathcal{P}_{phys}\hat{T}_K \tag{5.81}$$

  The same is true for $\hat{T}_V$. It is easy to see this by realizing that it is diagonal in the U basis and that the eigenvalues are gauge invariant.
- The transfer matrix is positive definite. We need to show that

$$\langle\psi|\hat{T}|\psi\rangle > 0 \tag{5.82}$$

  for all physical states $|\psi\rangle$ ($\langle\psi|\psi\rangle = 1$ and $\mathcal{P}_{phys}|\psi\rangle = |\psi\rangle$). Since

$$\langle\psi|e^{-\hat{V}}\hat{T}_K e^{-\hat{V}}|\psi\rangle = \langle\phi|\hat{T}_K|\phi\rangle \tag{5.83}$$

  where $|\phi\rangle \equiv e^{-\hat{V}}|\psi\rangle$, it is sufficient to prove the positivity of $\hat{T}_K^0$

$$\langle\psi|\hat{T}_K^0|\psi\rangle = e^{-\frac{2N}{g_0^2}} \int dU dU' \psi[U']^*\psi[U]\ \exp\left(\frac{1}{g_0^2}\sum_{k,\mathbf{x}}\mathrm{Tr}[U'_k(\mathbf{x})U_k^\dagger(\mathbf{x}) + h.c.]\right).$$

(5.84)

We can now expand the exponential in powers of $1/g_0^2$. Each term in the series can be shown to be positive by noticing that each term is a positive constant (coefficients of the Taylor expansion of the exponential) times an integral of the form

$$\int dU dU' \psi[U']^* \psi[U] \; (U'_{\alpha\beta} U^*_{\alpha\beta})^n (U_{\gamma\delta} U'^*_{\gamma\delta})^m$$
$$= r^*_{\alpha_1 \ldots \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots \gamma_m, \delta_1, \ldots \delta_m} r_{\alpha_1 \ldots \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots \gamma_m, \delta_1, \ldots \delta_m} \geq 0 \qquad (5.85)$$

where

$$r_{\alpha_1 \ldots \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots \gamma_m, \delta_1, \ldots \delta_m} \equiv \int dU \psi[U] U^*_{\alpha_1\beta_1} \ldots U^*_{\alpha_n\beta_n} U_{\gamma_1\delta_1} \ldots U_{\gamma_m\delta_m}. \qquad (5.86)$$

All terms must vanish for it to be zero. If the integral of a function $\psi[U]$ with any power of U and U^* is zero, the function must vanish. Therefore, for all normalizable wave functions, the positivity condition must hold.

Summarizing, we have identified the field operators, $\hat{U}_k(\mathbf{x})$, which represent the spatial links at $x_0 = 0$, and a positive transfer operator that determines their time evolution. Euclidean correlation functions of gauge-invariant combinations of such operators at arbitrary times can be represented by the corresponding functional integrals in the Wilson formulation of lattice gauge theories. The simplest gauge-invariant field operator is the spatial plaquette.

Having a unitary theory is reassuring, but the infrared behaviour of this theory is highly non-trivial. We believe two fundamental phenomena take place:

- Generation of a mass gap (in spite of the absence of dimensionful couplings)
- Confinement or the property that asymptotic states are gauge singlets

A very useful intuition can be obtained from the strong coupling expansion of the lattice theory, as first realized by Wilson (Wilson, 1974), where both phenomena can be shown to take place.

5.4 Strong Coupling Expansion: confinement, mass gap

The strong coupling expansion is an expansion in inverse powers of the coupling g_0, which by the structure of the path integral is equivalent to a high temperature expansion of the statistical system:

$$\mathcal{Z} = C \int \prod_l dU_l e^{-\frac{\beta}{2N} \sum_p \left[\chi(U_p) + \chi(U_p^\dagger)\right]}, \qquad (5.87)$$

where l, p is a short-hand notation for the links and plaquettes respectively,

$$\chi(U_p) \equiv \mathrm{Tr}\,[U_p] \qquad (5.88)$$

is the character of U_p and

$$\beta \equiv \frac{2N}{g_0^2}. \tag{5.89}$$

Therefore a series expansion for large g_0 corresponds to an expansion for small β or high temperature:

$$\mathcal{Z} = \int \prod_l dU_l \prod_p \sum_n \frac{1}{n!} \left(\frac{\beta}{2N}\right)^n (\chi(U_p) + \chi(U_p)^*)^n . \tag{5.90}$$

Since $\chi(U_p)$ is bounded, the series has a finite radius of convergence, in contrast with the small g_0 expansion. The strong coupling expansion has been worked out to very high orders. A more detailed discussion can be found in the literature (Montvay and Münster, 1994).

Working out the leading contribution to a given observable is quite simple noticing two facts. The leading order contribution has the lowest number of plaquettes. All link variables must be shared by at least two plaquettes, since any unpaired link results in a zero contribution by

$$\int dU \; U_{\alpha\beta} = 0. \tag{5.91}$$

For the following two examples, the only non-trivial integral needed is that of two links

$$\int dU \; U_{\alpha\beta} U^\dagger_{\gamma\delta} = \frac{1}{N} \delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{5.92}$$

5.4.1 Plaquette-Plaquette correlator and mass gap

We have seen that correlation functions of spatial plaquettes should be able to describe the propagation and scattering of physical particles. Since these objects are gauge invariant, they cannot be gluons and they are called generically *glueballs*. According to the Källen-Lehmann representation, we should be able to find out the presence of a mass gap in the theory by studying the correlator of two spatial plaquettes at large time separation.

Let us consider a plaquette $U_{kl}(\mathbf{x}, x_0)$ in any two spatial directions, $\hat{k}$ and $\hat{l}$, fixed at a position $(\mathbf{x}, x_0)$ and another one parallel and with opposite orientation to the first at the position $(\mathbf{x}, x_0 + T)$. The leading diagram with paired links and the minimum tiling of plaquettes is given by a rectangle linking the two external plaquettes, Fig. 5.2. The β and N dependence is given by

$$\left(\frac{\beta}{2N}\right)^{N_p} \left(\frac{1}{N}\right)^{N_i} N^{N_v}, \tag{5.93}$$

since each internal plaquette brings a factor $\beta/2N$, each integral over two paired links brings in a factor $1/N$, eq. (5.92), and each vertex gives a factor of N. In this case we have

$$N_p = \#plaquettes = 4T/a \tag{5.94}$$

$$N_i = \#integrals = \#links/2 = 2(N_p + 2) \tag{5.95}$$

$$N_v = \#vertices = N_v = 4(T/a + 1) \tag{5.96}$$

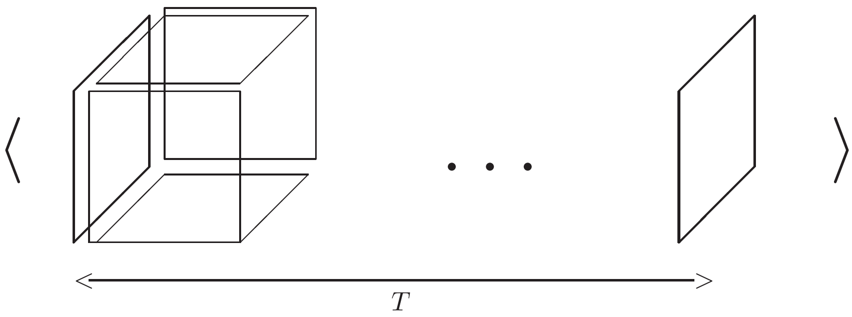


Fig. 5.2 Minimum tiling of the plaquette two-point correlator

Putting all together we find

$$C_{pp}(T) \sim \left(\frac{\beta}{2N^2}\right)^{4T/a} = \exp\left(-\frac{4}{a}\ln\left(\frac{2N^2}{\beta}\right)T\right), \tag{5.97}$$

therefore the correlator decays exponentially in time as expected in a theory with a finite mass gap, where correlators decay as $\exp(-mT)$. In this case the mass gap is

$$m \sim \frac{4}{a}\ln\left(\frac{2N^2}{\beta}\right). \tag{5.98}$$

Unfortunately no continuum limit can be reached in the strong coupling expansion since $\lim_{a\to 0} ma =$ finite. It is only in the continuum limit where we expect to find the universal behaviour of Yang-Mills field theory and therefore this result is not enough to prove the existence of a mass gap. One would need to ensure by other means that this behaviour survives in the continuum limit.

5.4.2 Wilson Loop and the static potential

Let us consider a rectangular loop with two spatial sides and two temporal ones, W_{RT}, Fig. 5.3. The spatial side length is R and the temporal one is T. It is easy to work out the leading order strong coupling behaviour of such an observable. It corresponds to the diagram where the loop is tiled up with plaquettes parallel to the loop. The behaviour is

$$\langle W_{RT}\rangle = \left(\frac{\beta}{2N}\right)^{N_p}\left(\frac{1}{N}\right)^{N_i} N^{N_v}, \quad N > 2 \tag{5.99}$$

where it is easy to count plaquettes, paired links and vertices:

$$N_p = (R/a)(T/a) \quad N_i = 2N_p + (R/a + T/a) \quad N_v = (R/a+1)(T/a+1), \tag{5.100}$$

so the final result is

$$\langle W_{RT}\rangle \sim N\left(\frac{\beta}{2N^2}\right)^{RT/a^2} \sim \exp\left(-\ln\left(\frac{2N^2}{\beta}\right)\frac{RT}{a^2}\right) \sim \exp\left(-\sigma \text{Area}\right). \tag{5.101}$$

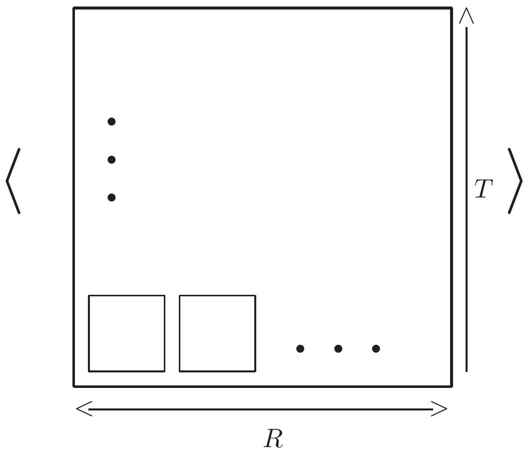


Fig. 5.3 Minimum tiling of the Wilson loop, W_{RT}.

Therefore the rate of the exponential decay as the temporal extent increases goes with the area encircled by the Wilson loop. This behaviour is called *area-law* and is a criterium for confinement. We now discuss why this is so.

The Wilson loop is related to the *static potential*, that is the potential of two point sources infinitely heavy and separated by a distance R. Let us consider for simplicity the case of scalar particles (the result will not depend on the spin). The static limit corresponds to an action where the spatial derivatives (spatial momenta) are neglected:

$$S_{stat}[\phi] = a^4 \sum_x \frac{1}{2} \left[\left(\hat{\partial}_0 \phi \right)^* \hat{\partial}_0 \phi + m^2 |\phi|^2 \right], \qquad |\hat{\partial}_k \phi| \ll m\phi, \tag{5.102}$$

the field values at different space points $\mathbf{x}$ are independent variables.

One can show (see exercise) that the correlator in the static approximation and in the presence of a background gauge field is

$$\langle \phi(\mathbf{x}, x_0) \phi^\dagger(\mathbf{y}, y_0) \rangle_\phi = \frac{a}{2\sinh(a\omega)} e^{-(x_0 - y_0)\omega} \delta(\mathbf{x} - \mathbf{y}) U(\mathbf{x}, x_0; \mathbf{y}, y_0), \tag{5.103}$$

where $U(\mathbf{x}, x_0; \mathbf{y}, y_0)$ is the parallel transporter and

$$\cosh(a\omega) = 1 + \frac{1}{2} a^2 m^2. \tag{5.104}$$

Exercise 4.4 Prove eq. (5.103) in the absence of gauge fields, ie. with the parallel transporter set to the identity. Show that in the presence of gauge fields the static propagator eq. (5.103) satisfies

$$(-\nabla_0^*\nabla_0 + m^2)\langle\phi(x)\phi^\dagger(y)\rangle_\phi = \delta(x-y). \tag{5.105}$$

The simplest gauge-invariant operator representing a quark and antiquark separated by some spatial distance $|\mathbf{y}-\mathbf{x}| = R$ at time t is

$$\mathcal{O}(t) = \phi^\dagger(\mathbf{y},t)U(\mathbf{y},t;\mathbf{x},t)\phi(\mathbf{x},t). \tag{5.106}$$

The correlator at large times $T \to \infty$,

$$C_{q\bar{q}}(T) \equiv \langle\mathcal{O}^\dagger(T)\mathcal{O}(0)\rangle_{\phi,U} \tag{5.107}$$

represents a quark-antiquark pair separated by a distance R that is created at time $x_0 = 0$ and evolves until time T. Integrating over the scalar fields, using eq. (5.103), and neglecting factors that do not depend on R (e.g $\exp(-T\omega)$) we get

$$C_{q\bar{q}}(T) \sim \langle\mathrm{Tr}[U(\mathbf{y},T;\mathbf{y},0)U(\mathbf{y},0;\mathbf{x},0)U(\mathbf{x},0;\mathbf{x},T)U^\dagger(\mathbf{y},T;\mathbf{x},T)]\rangle_U = \langle W_{RT}\rangle, \tag{5.108}$$

that is, the R dependence of this correlator is the same as that of a Wilson loop of area RT. We expect therefore that the exponential decay in time of such correlator gives us information about the energy of this system. The energy will contain a R-independent contribution, but it will also depend on the distance due to the potential energy between the quark and antiquark. We therefore expect

$$C_{q\bar{q}}(T) \sim \exp(-E(R)T), \tag{5.109}$$

where

$$E(R) = E_0 + V(R). \tag{5.110}$$

Relating eqs. (5.101) and eq. (5.108), we have

$$\lim_{\beta\to 0} V(R) = \frac{R}{a^2}\ln\left(\frac{2N^2}{\beta}\right) + ... = \sigma R + ..., \tag{5.111}$$

where σ is called *the string tension*:

$$\lim_{\beta\to 0}\sigma = \frac{1}{a^2}\ln\left(\frac{2N^2}{\beta}\right). \tag{5.112}$$

The linear behaviour of the potential as a function of R is a criterium for confinement, because the potential energy grows without bound when the quark and the antiquark are pulled apart.

Unfortunately, once more, the finite string tension that we find in the strong coupling limit does not imply that there is one in the continuum limit because

$$\lim_{a\to 0} a^2\sigma = \text{finite}, \tag{5.113}$$

and therefore σ diverges in the continuum limit.

These two simple examples show that the strong coupling analysis gets all the qualitative behaviour right, but there is no continuum limit in this approximation. We will see that a continuum limit can be shown to exist in the opposite extreme of small coupling, as expected from perturbative renormalizability.

5.5 Weak coupling expansion

Perturbatively we know that Yang-Mills theories are renormalizable and this, according to Wilson's renormalization group, implies that a continuum limit can be defined in lattice perturbation theory.

On the lattice, the weak coupling expansion corresponds to a saddle-point expansion around the configurations with vanishing action. We have seen that these correspond to all plaquettes being the identity:

$$U_p = 1. \tag{5.114}$$

These in turn are pure gauge configurations that are gauge equivalent to the configuration with all links set to the identity:

$$U_\mu(x) = 1. \tag{5.115}$$

Near this configuration, a convenient parametrization of the link variables is the exponential mapping

$$U_\mu(x) = \exp\left(-ig_0 a T^a A^a_\mu(x)\right), \tag{5.116}$$

but it is necessary to fix the gauge if we are going to integrate over unbounded gauge fields A^a_μ, just as in the continuum.

5.5.1 Gauge Fixing

The gauge fixing procedure on the lattice follows closely that in the continuum.

1) Choose a gauge fixing condition such as

$$G[U] = 0. \tag{5.117}$$

The gauge-fixing functional $G[U]$ is a function of the link variables and it is well defined for U near the identity. It must also satisfy that for any U near the identify, there is one and only one gauge transformation g such that

$$G[U^g] = 0. \tag{5.118}$$

2) Include the gauge fixing in the path integral by

$$\int dUe^{-S[U]} = \int d\Omega \int dUe^{-S[U]}\delta(G[U^{\Omega}])\Delta[U], \tag{5.119}$$

where

$$\Delta[U]^{-1} \equiv \int d\Omega\delta(G[U^{\Omega}]), \tag{5.120}$$

which is gauge invariant. Using the invariance of the measure, dU, it is easy to see that the integrand of eq. (5.119) does not depend on Ω and since the integral $\int d\Omega = 1$, we have

$$\int dUe^{-S[U]} = \int dUe^{-S[U]}\delta(G[U])\Delta[U]. \tag{5.121}$$

3) Rewrite the operator $\Delta[U]$ as a local ghost contribution using the Faddeev-Popov trick (Peskin and Schroeder, 1995). For any configuration U, let $\Omega_0(U)$ be the gauge transformation that satisfies

$$G[U^{\Omega_0(U)}] = 0. \tag{5.122}$$

Consider an infinitesimal gauge transformation, $\Omega_\epsilon = \exp(i\epsilon_a T^a)$:

$$G[U^{\Omega_0(U)\Omega_\epsilon}] = G[U^{\Omega_0(U)}] + \left.\frac{\partial G^a[U^{\Omega_0(U)\Omega_\epsilon}]}{\partial \epsilon_b}\right|_{\epsilon=0} \epsilon_b + ... \equiv M_{ab}[U]\epsilon_b + ... \tag{5.123}$$

If $\det(M[U]) \neq 0$, we can restrict the Ω integration in eq. (5.120) to the neighbourhood of Ω_0

$$\Delta[U]^{-1} = \int d\Omega\delta(G[U^{\Omega}]) = \int d\epsilon\delta(M[U]\epsilon) = \frac{1}{\det(M[U])}. \tag{5.124}$$

The determinant can now be included as an integral over Grassmann variables, or ghost fields

$$\Delta[U] = \int d\bar{c}dc\; e^{-S_{FP}[c,\bar{c},U]}, \quad S_{FP}[c,\bar{c},U] \equiv \bar{c}^a M_{ab}[U]c^c. \tag{5.125}$$

This the Faddeev-Popov term.

4) Rewrite the delta function as a Gaussian integral.
Consider a different gauge fixing functional $G'[U] = G[U] + k^a T^a$, with k^a some constants. Since $M'[U] = M[U]$, $\Delta[U]$ is the same and the partition function does not depend on k^a. We can therefore integrate over them with a gaussian weight

$$\begin{aligned}\mathcal{Z} &\sim \int \prod_a dk_a e^{-\frac{1}{2\alpha}\sum_a k_a^2} \int d\bar{c}dcdU e^{-S[U]-S_{FP}[c,\bar{c},U]}\; \delta(G(U)+k) \\ &= \int d\bar{c}dcdU\; e^{-S[U]-S_{FP}[c,\bar{c},U]-\frac{1}{2\alpha}\sum_a G^a(U)G^a(U)}.\end{aligned} \tag{5.126}$$

This is the starting point of lattice perturbation theory.

A commonly used gauge is the *Lorentz gauge*:

$$G[U] = \sum_{\mu} \hat{\partial}^{*}_{\mu} A_{\mu}(x), \tag{5.127}$$

where the field A_μ is defined via the exponential mapping eq. (5.116).

The last ingredient that we should specify is the measure for the exponential mapping. It can be shown that the Haar measure for each of the link variables can be written as

$$dU = \prod_{x,\mu} dU_{\mu}(x) = \prod_{x,\mu} \exp\left(-\mathrm{Tr}\left[\ln\left(\frac{2}{\omega}\sinh\left(\frac{\omega}{2}\right)\right)\right]\right) dA_{\mu} \tag{5.128}$$

with

$$\omega(U)_{ab} \equiv g_0 f_{abc} A^{c}_{\mu}(x), \tag{5.129}$$

and f_{abc} are the structure constants of the group, satisfying

$$[T^a, T^b] = i f_{abc} T^c. \tag{5.130}$$

Exercise 4.5 Show the following properties for $SU(N)$.

- Let $U(\alpha) \equiv \exp(i\alpha^a T^a)$. For λ a real number, define

$$R(\lambda)_{ab} \equiv 2\mathrm{Tr}\left[U(\lambda\alpha) T^a U^{\dagger}(\lambda\alpha) T^b\right]. \tag{5.131}$$

  Show that $R(\lambda)$ satisfies the following differential equation

$$\frac{\partial R}{\partial \lambda} = -i\hat{\alpha} R \quad \hat{\alpha} \equiv \alpha_a t^a \quad (t^a)_{bc} = -i f_{abc}, \tag{5.132}$$

  and therefore that $R(1) = \exp(-i\hat{\alpha})$.
- Next define

$$M(\lambda) \equiv U(\lambda\alpha) U^{\dagger}(\lambda(\alpha + \epsilon)). \tag{5.133}$$

  Neglecting terms of $\mathcal{O}(\epsilon^2)$ show that

$$\frac{\partial M}{\partial \lambda} = -i\epsilon^a R_{ab}(\lambda) T^b + \mathcal{O}(\epsilon^2), \tag{5.134}$$

  and that this implies

$$M(1) = 1 - i\epsilon^a \left(\frac{1 - \epsilon^{-i\hat{\alpha}}}{i\hat{\alpha}}\right)_{ab} T^b + \ldots \tag{5.135}$$

- Show that the previous results imply ($X, Y \in su(n)$):

$$(e^{iX} Y e^{-iX})_a = Y^b (e^{-i\hat{X}})_{ba} \tag{5.136}$$

$$e^{iX} \partial_a e^{-iX} = -i \left(\frac{1 - e^{-i\hat{X}}}{i\hat{X}} \right)_{ab} T^b. \tag{5.137}$$

- Use these properties to show that the Haar measure for the exponential coordinates, α_a, can be written as

$$dU = \exp\left(\mathrm{Tr}\left[\ln\left(\frac{2}{\hat{\alpha}} \sin(\hat{\alpha}/2) \right) \right] \right) \prod_a d\alpha_a \tag{5.138}$$

- Show that in $SU(N)$, for $U_\mu = \exp(-ig_0 A_\mu)$ and for the Lorentz gauge

$$G[U] = g_0 \sum_\mu \hat{\partial}^*_\mu A_\mu(x) \tag{5.139}$$

  the operator M that enters in the ghost action is

$$M(U) = \hat{\partial}^*_\mu \left\{ \frac{ig_0 \hat{A}_\mu}{(1 - \exp(+ig_0 \hat{A}_\mu)} \hat{\partial}_\mu - ig_0 \hat{A}_\mu \right\} \tag{5.140}$$

  where $\hat{A}_\mu \equiv A^a_\mu t^a$.

5.5.2 Feynman rules

The derivation of the Feynman rules for the gauge-fixed lattice action

$$S_{GF}[c, \bar{c}, U] = S[U] + S_{FP}[c, \bar{c}, U] + \frac{1}{2\alpha} \sum_a G^a(U) G^a(U) \tag{5.141}$$

is conceptually straightforward but quite complicated! As usual we go over to momentum space and assign the gauge potential, defined from the exponential map, eq. (5.116) to the points in the middle of the link $x + \hat{\mu}a/2$:

$$A_\mu(p) = a^4 \sum_x e^{ip(x + a\frac{\hat{\mu}}{2})} A_\mu(x). \tag{5.142}$$

The leading contribution $\mathcal{O}(g_0^0)$ is only quadratic in the fields. The gauge part is:

$$S^{(0)}[U] = \frac{1}{2} \int_{BZ} \frac{d^4k}{(2\pi)^4} A^a_\mu(-k) e^{\frac{ik_\mu a}{2}} \left(\delta_{\mu\nu} \hat{k}^2 - (1 - \alpha) \hat{k}_\mu \hat{k}_\nu \right) e^{\frac{-ik_\nu a}{2}} A_\nu(k), \tag{5.143}$$

with

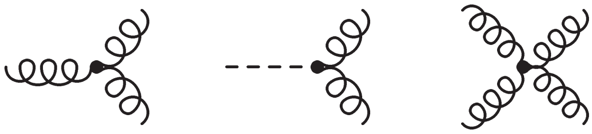
$$\hat{k}_\mu = \frac{2}{a} \sin\left(\frac{k_\mu a}{2} \right) \qquad \hat{k}^2 = \sum_\mu \hat{k}^2_\mu. \tag{5.144}$$

The corresponding Feynman rules for the gauge and ghost propagators are:

$$-\frac{\delta_{ab}}{\hat{k}^2}\left[\delta_{\mu\nu} - (1-\alpha)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2}\right],$$

$$\frac{\delta_{ab}}{\hat{k}^2}.$$

At higher order in g_0 there are diagrams that have continuum analogs, such as:



,

and some more that do not appear in the continuum, such as a gluon mass term or a two-gluon-two-ghost vertex:

$$-(2\pi)^4 \frac{N g_0^2}{12 a^2}\ \delta_{ab}\delta_{\mu\nu},$$

For more details see (Rothe, 2005).

At one loop one can explicitely check renormalizability. By power-counting we expect that only the 2, 3 and 4 vertex functions are divergent in the continuum limit and one can show that the divergence in $\Gamma^{(2)}_{\mu\nu;ab}$ can be reabsorbed in a field redefinition

$$Z^{1/2} A_{\mu R} = A_\mu, \tag{5.145}$$

while the divergences in 3 and 4-point vertex functions are related by gauge-invariance and can be reabsorbed in a redefinition of the coupling

$$Z_g g_R = g_0. \tag{5.146}$$

A large number of seemingly miraculous cancellations take place, ensuring the absence of disastrous contributions, such as a gluon mass term or Lorentz-non-invariant counterterms. The miracle can be shown to occur to all orders by realizing that there is an exact BRST invariance satisfied also on the lattice. I refer to the lectures of P. Weisz for further details (Weisz, 2009).

There seems to be a continuum limit in perturbation theory, but we still do not know if this perturbative analysis provides a solid proof of renormalizability.

We can look at the Callan-Symanzik equations to understand the gauge coupling flow as we approach the continuum limit. Let us consider the momentum subtraction scheme:

$$\begin{aligned}\Gamma^{(2)}(k)|_{k^2=\mu^2} &= \mathrm{tree-level}\\ \Gamma^{(4)}(k_1,k_2,k_3)|_{k_i k_j = \frac{1}{2}(3\delta_{ij}-1)\mu^2} &= \mathrm{tree-level},\end{aligned} \tag{5.147}$$

where $\mu a \ll 1$. From these two conditions the renormalized coupling is found to be:

$$g_R^2(\mu) = g_0^2 \left(1 - \frac{g_0^2}{16\pi^2}\frac{11N_c}{3}(\ln(a^2\mu^2) + c')\right). \tag{5.148}$$

As we approach the continuum limit we must tune the coupling $g_0(a)$ so as to keep the physical coupling fixed. Neglecting scaling violations $\mathcal{O}(\mu a)$ we have the following RG equation:

$$\beta(g_0) \equiv -a \left.\frac{\partial g_0}{\partial a}\right|_{g_R \text{ fixed}} = -\beta_0 g_0^3 - \beta_1 g_0^5 + ... \tag{5.149}$$

where β_0, β_1 are universal (ie. do not depend on the regularization scheme). For $SU(3)$:

$$\beta_0 = \frac{11}{16\pi^2} > 0, \quad \beta_1 = \frac{102}{(16\pi^2)^2}. \tag{5.150}$$

This equation shows that g_0 decreases as we approach the continuum limit, so perturbation theory becomes more accurate as we approach this limit. In fact $g_0 = 0$ is a zero of the β function, i.e. an *UV fixed point*, therefore our target continuum limit corresponds to $g_0 = 0$. Therefore the perturbative analysis of renormalizability is perfectly justified. Note that this would not be the case if the fixed-point would occur at a different value of g_0.

We can integrate the RG equation to get

$$a = c \exp\left(\frac{-1}{2\beta_0 g_0^2}\right)(g_0^2)^{-\frac{\beta_1}{2\beta_0^2}}, \tag{5.151}$$

where c is a constant of integration and does not depend on a, even though it has the same dimensions. It is common practice to define a Λ parameter in terms of this constant

$$a\Lambda \equiv \exp\left(\frac{-1}{2\beta_0 g_0^2}\right)(\beta_0 g_0^2)^{\frac{-\beta_1}{2\beta_0^2}}, \tag{5.152}$$

which remains constant in the continuum limit. All scales should be proportional to Λ as we approach the continuum limit. It can therefore be taken as a reference scale, although it should always be remembered that this scale depends on the regularization scheme.

5.6 Topological charge

$SU(3)$ gauge fields in the continuum fall into distinct topological sectors labelled by the integer

$$Q = -\frac{1}{32\pi^2}\int d^4x \; \epsilon_{\mu\nu\rho\sigma} \; \mathrm{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right], \tag{5.153}$$

called *topological charge or instanton number*.

That configurations with $Q \neq 0$ exist and are important for physics was understood when classical (i.e. finite action) solutions where found with $Q = 1$ (instantons) (Belavin, Polyakov, Schwartz and Tyupkin, 1975). They are believed to play an important role ('t Hooft, 1976; Witten, 1979; Veneziano, 1979) in several fundamental problems such as the

- $U_A(1)$ problem in QCD
- Violation of $B + L$ in the Standard Model (where B is baryon number and L is lepton number)

Consider a continuum gauge field in a box of size $T \times L^3$, with the following boundary conditions (periodicity up to a gauge transformation):

$$\begin{aligned} A_\mu(x)|_{x_i=L} &= A_\mu(x)|_{x_i=0}\,, \quad \hat{k} = 1,2,3 \\ A_\mu(x)|_{x_0=T} &= \Omega(\mathbf{x}) A_\mu(x)|_{x_0=0} \Omega^\dagger(\mathbf{x}) + i\Omega(\mathbf{x})\partial_\mu \Omega(\mathbf{x})^\dagger, \end{aligned} \tag{5.154}$$

where $\Omega(\mathbf{x})$ is periodic in the spatial directions, that is, it is map of the 3-torus on $SU(3)$. Such functions fall in homotopy classes characterized by the winding number of the map which coincides with Q.

If $Q \neq 0$, no smooth gauge transformation $g(x)$ exists such that

$$g(0, \mathbf{x}) = I, \qquad g(T, \mathbf{x}) = \Omega(\mathbf{x}), \tag{5.155}$$

Otherwise, the winding could be gauged away, which must not be possible, since Q is gauge invariant.

Example: Let us consider the simpler case of $U(1)$ in 2D. Let us consider the function Ω that maps the circle, T^1 into $U(1)$:

$$\Omega : T_1 \to U(1) \tag{5.156}$$

$$x \to e^{i2\pi \frac{x}{L} q} \tag{5.157}$$

The topological charge in 2D is given by

$$Q = -\frac{1}{2\pi} \int d^2x \sum_{\mu<\nu} \epsilon_{\mu\nu} F_{\mu\nu} = \frac{i}{2\pi} \int_0^L dx \Omega \partial_x \Omega^\dagger = q, \tag{5.158}$$

where we have used eq. (5.154).

Exercise 4.6 A geometrical definition of topological charge in compact $U(1)$ in two dimensions. Consider the following local quantity

$$q_n = \frac{-i}{2\pi} \sum_{\mu<\nu} \epsilon_{\mu\nu} \ln U_{\mu\nu}(n), \tag{5.159}$$

and its global sum:

$$Q = \sum_n q_n \tag{5.160}$$

in a periodic lattice. Check that q_n is gauge invariant. Show that Q is an integer. Show that its naive continuum limit is the 2D topological charge.

Is there topology in the lattice formulation ? If for example we have the lattice boundary conditions

$$U_k(x)|_{x_0=T} = \Omega(\mathbf{x})\; U_k(x)|_{x_0=0}\, \Omega(\mathbf{x} + a\hat{k})^\dagger, \tag{5.161}$$

it is obvious that Ω can be gauged away by changing the temporal link variables at the border as

$$U_0'(\mathbf{x}, T-a) = U_0(\mathbf{x}, T-a)\Omega^\dagger(\mathbf{x}), \tag{5.162}$$

which does not change the measure $dU_0' = dU_0$. Does this mean that we are missing $Q \neq 0$ configurations on the lattice ?

No, it means that all topological sectors correspond to periodic boundary conditions, and that all are connected by lattice gauge transformations. However, the configurations at the boundaries between the different topological sectors do not have a continuum limit, i.e. they become singular in the continuum limit, so there is no contradiction.

For some purposes it might be useful to define an integer topological charge also at finite a. This can be done in various ways, for example:

- Geometrical definition (Lüscher, 1982). A local density $q(x)$ can be defined with the following properties

$$\sum_x q(x) \in Z \qquad \lim_{a\to 0} q(x) = -\frac{1}{32\pi^2}\int d^4x \mathrm{Tr}\left[\tilde{F}_{\mu\nu}F_{\mu\nu}\right]. \tag{5.163}$$

  One can also show that Q does not change if a gauge-invariant constraint is set on the plaquettes $\mathrm{Tr}[U_p] \geq 1 - \epsilon$.
- Fermionic zero modes. The index theorem (Atiyah and Singer, 1971) establishes the following relation:

$$Q = n_R - n_L, \tag{5.164}$$

  where $n_{R/L}$ are the righ-handed/left-handed zero modes of the Dirac operator on the background gauge configuration with charge Q. Lattice fermions that satisfy a chiral symmetry at finite a such as Ginsparg-Wilson fermions allow to define a topological charge from the number of zero modes (Hasenfratz, Laliena and Niedermayer, 1998).

6

Lattice QCD

The original investigation on lattice field theory was motivated by the need to make predictions in QCD. There is by now little doubt that QCD is the right theory of the strong interactions. It is an $SU(3)$ gauge theory, with six flavours of quarks in the fundamental representation. The Euclidean action in the continuum is

$$S_{QCD} = \int d^4x \sum_q \bar{\psi}_q(\gamma_\mu D_\mu + m_q)\psi_q - \frac{1}{2g_0^2}\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}], \tag{6.1}$$

which has therefore seven free parameters: the gauge coupling and six quark masses.

Let us review briefly the main properties of QCD.

- Symmetries. At the classical level the symmetries of this action are
  - Lorentz invariance
  - $SU(3)$ gauge invariance
  - Discrete symmetries: C, P and T
  - Quark number: $\psi_q \to e^{i\alpha_q}\psi_q$

  In the absence of quark masses, there is a much larger global symmetry group which is a chiral $U(6)_L \times U(6)_R$:

  $$P_R\psi \to U_R P_R\psi \quad P_L\psi \to U_L P_L\psi \quad U_R, U_L \in U(6), \tag{6.2}$$

  where $\psi = (\psi_u, \psi_d, \psi_d, ..)$.
- Spontaneous chiral symmetry breaking

  The chiral flavour group is believed to be broken to $U(6)_V$ spontaneously by a quark condensate

  $$-\langle \bar{\psi}_i \psi_j \rangle \neq \Sigma\delta_{ij}, \tag{6.3}$$

  which is only invariant under $U_R = U_L = U_V$.
- Anomalous breaking of $U_A(1)$

  The $U(1)_A$ is broken by a different mechanism: via an anomaly. Indeed, even if there is no symmetry breaking by the vacuum or by the explicit mass terms, the current associated with this symmetry is not conserved. According to the Noether theorem, the axial current

  $$J_\mu^5 = \sum_q \bar{\psi}_q \gamma_\mu \gamma_5 \psi_q \tag{6.4}$$

should be conserved. However a one loop computation (Adler and Bardeen, 1969) shows that

$$\partial_\mu J^5_\mu = \frac{g_0^2}{16\pi^2}\epsilon_{\alpha\beta\gamma\delta}\mathrm{Tr}[F_{\alpha\beta}F_{\gamma\delta}], \tag{6.5}$$

while the vector current is of course conserved. We identify the topological charge on the right-hand side!

The spontaneous breaking of a global symmetry implies the presence of as many Nambu-Goldstone (Nambu, 1960; Goldstone, 1961) massless particles, as broken generators: the generators of the $SU(6)$ axial rotations (since $U_A(1)$ is not broken spontaneously).

In reality quark masses are not zero. They are plotted in a logarithmic scale in Fig. 6.1, where we see that they encompass five orders of magnitude. Compared to the mass gap of the pure gauge theory, $\sim 1GeV$, there are three quarks: u, d, s that can be considered light, while other three c, b, t are heavy. Therefore the approximate flavour symmetry in the presence of quark masses is at most $SU(3)$ and not $SU(6)$. We expect therefore that the spontaneous symmetry breaking results in eight lighter states corresponding to the Nambu-Goldstone bosons, which have the quantum numbers of the pseudoscalar mesons. Indeed the lightest excitations in QCD are the octet of pseudoscalar mesons: $\pi^\pm, \pi_0, K^\pm, K_0, \bar{K}_0, \eta$. The η', being the mode associated to $U(1)_A$, is significantly more massive, because it is not a Nambu-Goldstone boson.

Witten and Veneziano (Witten, 1979; Veneziano, 1979) got a prediction for the mass of this special meson in the large N_c limit:

$$\frac{F_\pi^2 m_{\eta'}^2}{2N_f} = \chi_{top} \equiv \int d^4x \langle Q(x)Q(0)\rangle, Q \equiv \frac{g_0^2}{32\pi^2}\epsilon_{\alpha\beta\gamma\delta}\mathrm{Tr}[F_{\alpha\beta}F_{\gamma\delta}]. \tag{6.6}$$

The η' mass can then be determined from a purely gauge observable, such as the topological susceptibility, which in the large N_c limit can be determined in the pure gauge theory!

QCD is a renormalizable theory, but perturbation theory does not provide a good description of its phenomenology at large distances or low energies, because the theory is strongly interacting. Indeed the main features of QCD that determine to a large extent its phenomenology are intrinsically non-perturbative: mass gap, confinement, spontaneous chiral symmetry breaking, anomalous currents, etc. Obviously the goal of a non-perturbative approach to QCD would be to understand from first principles all these phenomena, and to provide an accurate description of QCD phenomenology, such as the hadron spectrum and other properties.

Even though in most cases these would be postdictions, it is nevertheless extremely important to finalize with success this longterm project for several reasons:

- The flavour sector of the SM is poorly understood, and it is rather generic that models beyond the SM induce non-standard effects in flavour violating processes in the quark sector. Having precise predictions in the SM is therefore indispensable to search for such non-standard effects.

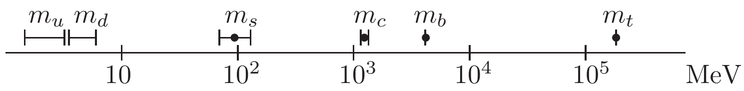


Fig. 6.1 Quark masses.

- It would allow to study QCD at high density and temperature, conditions of the early universe and of very dense systems such as neutron stars, that are not easy to reproduce in the laboratory. We refer to the lectures by U. Philipsen (Philipsen, 2009).
- QCD is in some sense a model field theory for many extensions of the SM, as well as for the lattice approach. In QCD we know where the UV fixed point lies so we know where the continuum limit is and how to approach it. The lattice method might be necessary to study other field theories, such as technicolor models or theories with dynamical gauge symmetry breaking, where things might not be as easy. Clearly having solved QCD is a benchmark to guide future investigations.

Giving the spread of quark masses that span six orders of magnitude, dealing with all quarks in a lattice simulation is very difficult since approaching the continuum limit in controlled conditions requires

$$am_q \ll 1, \tag{6.7}$$

and therefore extremely fine lattices. This brute force approach is not practical. Fortunately, when we try to describe the low energy regime, the effect of the heavy quarks can be accurately described by an effective theory that results from integrating them out. A consequence of the decoupling theorem (Appelquist and Carazzone, 1975) (which is another realization of Wilsonian renormalization group) is that the effects of the heavy quarks in the low-energy dynamics are well represented by local operators of the light fields only (gluons and the lighter quarks). The effect of the heavy scales is reabsorbed in the couplings. This implies that in order to study hadron processes at energies much lower than the heavy quark mass scales, we can simply ignore the heavy quarks.

We are also interested however in processes involving heavy hadrons. A way to do this is to consider them as static sources, as is done in the heavy quark effective theory. I refer to R. Sommer's lectures (Sommer, 2009) for a detailed discussion of this effective theory as an efficient tool to study heavy flavours on the lattice.

6.1 Wilson formulation of Lattice QCD

By now, it should be clear how to discretize this action following for example the Wilson approach

$$S_{QCD}[U,\bar{\psi},\psi] = S[U] + S_W[U,\bar{\psi},\psi] \tag{6.8}$$

where $S[U]$ is the plaquette action of eq. (5.39) and $S_W[U,\bar{\psi},\psi]$ is the Wilson action for each of the quark fields:

$$S_W[U,\bar{\psi},\psi] = a^4 \sum_{q,x} \bar{\psi}_q \left[D_W + m_q\right] \psi_q, \tag{6.9}$$

where the Wilson operator is

$$D_W \equiv \frac{1}{2}\left(\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - ar\nabla^*_\mu\nabla_\mu\right) \tag{6.10}$$

and

$$\begin{aligned}\nabla_\mu\psi(x) &= \frac{1}{a}\left[U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x)\right],\\ \nabla^*_\mu\psi(x) &= \frac{1}{a}\left[\psi(x) - U_\mu(x-a\hat{\mu})^\dagger\psi(x-\hat{\mu})\right].\end{aligned} \tag{6.11}$$

It is common practice to rewrite the fermionic action in terms of the parameter κ:

$$\begin{aligned}S_W = a^4 \Bigg\{ &\sum_{q,x} \bar{\psi}_q(x)\left[m_q + \frac{4r}{a}\right]\psi_q(x) + \frac{1}{2a}\sum_{q,x,\mu}\bar{\psi}_q(x)\left(\gamma_\mu - r\right)U_\mu(x)\psi_q(x+a\hat{\mu})\\ - \quad &\bar{\psi}_q(x)\left(\gamma_\mu + r\right)U^\dagger_\mu(x-a\hat{\mu})\psi_q(x-a\hat{\mu})\Bigg\}.\end{aligned} \tag{6.12}$$

The action can be rewritten as

$$\begin{aligned}S_W = a^4 \sum_{q,x}\bar{\psi}_q(x)\psi_q(x) - \kappa_q &\sum_{q,x,\mu}\Big(\bar{\psi}_q(x)(\gamma_\mu - r)U_\mu(x)\psi_q(x+a\hat{\mu})\\ + \quad &\bar{\psi}_q(x)(\gamma_\mu + r)U^\dagger_\mu(x-a\hat{\mu})\psi_q(x-a\hat{\mu})\Big),\end{aligned} \tag{6.13}$$

where we have introduced the kappa parameter:

$$\kappa_q \equiv \frac{1}{2am_q + 8r}. \tag{6.14}$$

In the free case, the massless limit corresponds to the critical value $\kappa_c = \frac{1}{8r}$.

The measures over the gauge links and the Grassmann variables are the same as defined before and therefore the partition function is

$$\mathcal{Z} = \int dU d\bar{\psi} d\psi e^{-S_{QCD}[U,\bar{\psi},\psi]} = \int dU \mathcal{Z}_F[U] e^{-S_g[U]} \tag{6.15}$$

where

$$\mathcal{Z}_F[U] \equiv \int d\bar{\psi} d\psi e^{-S_W[U,\bar{\psi},\psi]}. \tag{6.16}$$

Since the action is quadratic in the fermion fields, the integration over the Grassmann fields can be performed analytically giving

$$\mathcal{Z}_F[U] = \prod_q \det\left(D_W + m_q\right). \tag{6.17}$$

For sufficiently large m_q, the det() factors are positive, so they can be exponentiated to a real contribution to the gauge action. The integral over the gauge degrees of freedom can still be solved by importance-sampling methods. I refer to the lectures of M. Lüscher for more details(Lüscher, 2009).

The integration over Grassmann variables can always be done analytically for any correlation function involving fermion fields. For the quark propagator we have

$$\langle \psi_{\alpha,i}(x)\bar{\psi}_{\beta,j}(y)\rangle = \mathcal{Z}^{-1} \int DU \langle \psi_{\alpha,i}(x)\bar{\psi}_{\beta,j}(y)\rangle_F \prod_q \det\left(D_W + m_q\right)\ e^{-S_g[U]}, \tag{6.18}$$

where α, β and i, j are spin and flavour indices respectively, and

$$\langle \psi(x)_{\alpha i}\bar{\psi}(y)_{\beta j}\rangle_F = \mathcal{Z}_F{}^{-1} \int D[\bar{\psi}]D[\psi]\ \psi(x)_{\alpha i}\bar{\psi}(y)_{\beta j} e^{-S_F[U,\bar{\psi},\psi]} = \delta_{ij}\left[(D_W + m_i)^{-1}\right]^{\alpha\beta}_{xy}. \tag{6.19}$$

All fermion integrals result in products of propagators as expected from eq. (4.9).

6.1.1 Positivity of the transfer matrix and Hilbert space interpretation

The positivity of the transfer matrix, $\hat{T}$, can be proved from the results obtained for the gauge fields and the free fermions. Indeed the transfer matrix can be written as

$$\hat{T} = \hat{T}_F^{1/2}\hat{T}_g\hat{T}_F^{1/2}\hat{\mathcal{P}}_{phys}, \tag{6.20}$$

where $\hat{T}_F$ is the transfer matrix for fermions, eq. (4.83), coupled to the gauge fields in the temporal gauge. The positivity of $\hat{T}_F$ which can be proved in completely analogy with the free fermion case for $r = 1$. $\hat{T}_g$ is the transfer operator for gauge fields, eq. (5.76). The positivity of $\hat{T}$ follows from that of $\hat{T}_g$ and $\hat{T}_F$ (Lüscher, 1977).

6.1.2 Perturbative expansion, renormalization and continuum limit

The perturbative expansion can be worked out like in the pure gauge theory. The Feynman rules are supplemented by the fermion vertices with one, two and an arbitrary number of gluons. In the presence of fermions besides the 1PI divergent graphs we considered in the pure gauge case, there is also the fermion two-point vertex graph. A one loop computation shows that this divergence can be reabsorbed in a redefinition of the fermion mass and wave function, in agreement with the expectation of renormalizability.

On general grounds, we know that in order to warrant that we approach the continuum limit we should make sure that the symmetries are those of the QCD action. It is easy to check that the lattice action is invariant under C, P and T. It is also invariant under the flavour vector symmetries in the limit $m_q = 0$. However all axial symmetries are broken by the Wilson term. The question is then: what are the additional relevant or marginal operators that can appear in the continuum limit ? The only renormalizable operator that can be induced as a result of chiral symmetry breaking is of the form $\bar{\psi}\psi$. It is indeed a relevant operator which is generated with a coefficient $\frac{1}{a}$ and needs to be tuned, just as the mass of the scalar needed to be tuned to reach the critical line. Therefore the continuum limit of this action even for massless fermions requires a more complicated tuning:

$$g_0 \to 0, \qquad \kappa^q \to \kappa^q_c. \tag{6.21}$$

The theory is asymptotically free just like the pure gauge theory.

A very useful procedure to define the massless point, beyond perturbation theory, is to impose the PCAC relation.

6.1.3 Lattice symmetries and scaling violations

We have seen that the Wilson term breaks chiral symmetry, and in QCD the full chiral flavour symmetry group. This is in principle a disaster, because the low-energy properties of QCD depend in a strong way on the fact that this symmetry is broken only spontaneously as we discussed. It is therefore essential to make sure that the continuum limit is taken in such a way that QCD is recovered. The symmetries in the functional formalism result in a series of Ward-identities (WI), as we discussed in sec. 2.5. Therefore a way to ensure that the symmetry is recovered in the continuum limit is to ensure that renormalized Ward identities are satisfied up to terms that vanish in the continuum limit.

Bochicchio et al. studied for the first time how the chiral WI is recovered in the continuum limit of Wilson fermions (Bochicchio *et al.*, 1985). To derive the WI's we consider the following non-singlet transformation ($\mathrm{Tr}[T^a] = 0$)

$$\begin{aligned} \delta\psi(x) &\to i\epsilon_a(x) T^a \gamma_5 \psi(x), \\ \delta\bar{\psi}(x) &\to i\epsilon_a(x) \bar{\psi}(x) T^a \gamma_5. \end{aligned} \tag{6.22}$$

Performing such a change of variables in the expectation value of the operator O we get:

$$\langle \delta_\epsilon S_W \; O \rangle = \langle \delta_\epsilon O \rangle, \tag{6.23}$$

where

$$\delta_\epsilon S_W = a^4 \sum_x \epsilon^a(x) \left\{ i\bar{\psi}(x)\gamma_5\{M, T^a\}\psi(x) - i\sum_\mu \hat{\partial}^*_\mu A^a_\mu(x) + iX^a(x) \right\}, \tag{6.24}$$

and

$$X^a(x) = -\frac{r}{2a}\sum_\mu \; \left[\bar{\psi}(x)T^a\gamma_5 U_\mu(x)\psi(x+a\hat{\mu}) + \bar{\psi}(x)T^a\gamma_5 U^\dagger_\mu(x-a\hat{\mu})\psi(x-a\hat{\mu})\right.$$

$$+\bar{\psi}(x-a\hat{\mu})U_\mu(x-a\hat{\mu})T^a\gamma_5\psi(x)+\bar{\psi}(x+a\hat{\mu})U_\mu^\dagger(x)T^a\gamma_5\psi(x)$$

$$-\,4\bar{\psi}(x)T^a\gamma_5\psi(x)\big]=\delta_\epsilon(\text{Wilson term}). \tag{6.25}$$

and

$$A_\mu^a(x)\equiv\frac{1}{2}\left[\bar{\psi}(x)\gamma_\mu\gamma_5T^aU_\mu(x)\psi(x+a\hat{\mu})+\bar{\psi}(x+a\hat{\mu})\gamma_\mu\gamma_5T^aU_\mu^\dagger(x)\psi(x)\right]. \tag{6.26}$$

M is the quark mass matrix. In the naive continuum limit, we find that $X^a\to 0$, while $A_\mu(x)$ goes to the continuum axial current.

So the WI on the lattice reads:

$$\langle O(y)\hat{\partial}_\mu^* A_\mu^a(x)\rangle=\langle O(y)\bar{\psi}(x)\gamma_5\{M,T^a\}\psi(x)\rangle+\langle O(y)X^a(x)\rangle-i\left\langle\frac{\delta O(y)}{\delta\epsilon^a(x)}\right\rangle. \tag{6.27}$$

The anomalous term, X^a, even though vanishing in the naive continuum limit will generate divergences that need to be renormalized. Being a local operator of $d=5$ will generically mix with the operators $\hat{\partial}_\mu^* A_\mu$ and with the pseudoscalar density $P^a=\bar{\psi}(x)T^a\gamma_5\psi(x)$, so in general

$$X^a=-2\bar{m}P^a-(Z_A-1)\hat{\partial}_\mu^* A_\mu+X_R^a, \tag{6.28}$$

where the last term is a renormalized operator that vanishes in the continuum limit and $\bar{m}$ and Z_A-1 are the mixing coefficients of X^a with the lower dimensional operators. $\bar{m}\sim a^{-1}$, while Z_A can be shown to be finite. Therefore

$$\lim_{a\to 0}\ \langle O(y)Z_A\partial_\mu^* A_\mu^a\rangle=\lim_{a\to 0}\ \langle O(y)\bar{\psi}(x)\gamma_5\{M-\bar{m},T^a\}\psi(x)\rangle-i\left\langle\frac{\delta O(y)}{\delta\epsilon^a(x)}\right\rangle. \tag{6.29}$$

In the continuum limit we recover the standard chiral WI, with the lattice current normalized by Z_A and the quark mass is proportional to $M-\bar{m}$. In general the scaling violations are $O(a)$, however the improvement program described in (Weisz, 2009) allows to reach $O(a^2)$.

In summary, the consequence of the explicit chiral symmetry breaking by the Wilson term is twofold:

- The bare mass M needs to be tuned non-perturbatively to fix the quark mass, for example, the so-called PCAC quark mass can be obtained from the ratio (up to a multiplicative renormalization)

$$\frac{\langle\hat{\partial}_\mu^* A_\mu^a(x)P^a(0)\rangle}{\langle P^a(x)P^a(0)\rangle}\sim m_{PCAC}. \tag{6.30}$$

- The axial current is renormalized. For a method to determine Z_A non-perturbatively and further details on the uses of WIs see the lectures of P. Weisz (Weisz, 2009) and A. Vladikas(Vladikas, 2009).

Exercise 5.1 Show that the Wilson action for QCD has the following discrete symmetries

$$P : \psi(x) \to \gamma_0 \psi(x_P) \tag{6.31}$$
$$\bar{\psi}(x) \to \bar{\psi}(x_P)\gamma_0 \tag{6.32}$$
$$U_0 \to U_0(x_P) \tag{6.33}$$
$$U_k \to U_k^\dagger(x_P - a\hat{k}) \tag{6.34}$$

$$T : \psi(x) \to \gamma_0\gamma_5 \psi(x_T) \tag{6.35}$$
$$\bar{\psi}(x) \to \bar{\psi}(x_T)\gamma_5\gamma_0 \tag{6.36}$$
$$U_0 \to U_0^\dagger(x_T - a\hat{0}) \tag{6.37}$$
$$U_k \to U_k(x_T) \tag{6.38}$$

$$C : \psi(x) \to C\bar{\psi}^T(x) \tag{6.39}$$
$$\bar{\psi}(x) \to -\psi^T(x)C^{-1} \tag{6.40}$$
$$U_\mu \to U_\mu^* \tag{6.41}$$

where $x_P = (x_0, -\mathbf{x})$, $x_T = (-x_0, \mathbf{x})$ and $C = \gamma_0\gamma_2$, satisfying $C\gamma_\mu C = -\gamma_\mu^* = -\gamma_\mu^T$.
Exercise 5.2 Show that the Wilson action for QCD is invariant under global $U_V(N_f)$ in the quark mass degenerate limit

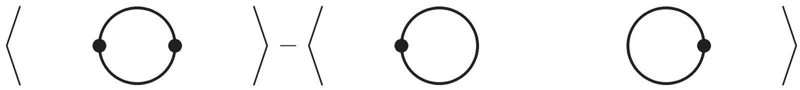
$$q \to Uq \quad \bar{q}_f \to \bar{q}U^\dagger \quad U \in U(N_f). \tag{6.42}$$

Derive the lattice WI for the $U_V(N_f)$ symmetry and identify the conserved vector current.

6.2 Observables

We will briefly discuss a few of the observables that are routinely measured in lattice QCD. The first important question is of course the low-lying spectrum. Computing the meson and baryon masses requires the computation of two-point correlators of appropriate operators. The Källen-Lehmann representation implies that the large time behaviour of these two point functions are dominated by the lightest one-particle states with the same quantum numbers.

How do we choose the operator ? We have seen, from the transfer matrix construction that operators with a Hilbert interpretations are products of the fundamental fields $\psi, \bar{\psi}$ and the spatial plaquettes at fixed times. In principle any operator in the Hilbert space can be represented by creation and annihilation operators that create the one-particle asymptotic states in the interacting theory. Ensuring that the quantum numbers are the right ones (spin, color, isospin, parity, etc) the operator will generically have an overlap with the one-particle states. Obviously we do not know a priori which operator maximizes this overlap and there are several techniques to improve it (variational techniques, smearing, etc), which we will not discuss here.



(6.47)

Fig. 6.2 Connected and disconnected contribution to a meson correlator

6.2.1 Mesons

The simplest operators that are used to compute meson correlation functions are of the form:

$$M^a(x) \equiv \bar{\psi}_{\alpha ic}(x)\Gamma_{\alpha\beta}T^a_{ij}\psi_{\beta jc}(x), \tag{6.43}$$

where

$$\Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, ...\} \tag{6.44}$$

for the scalar, axial, vector and axial vector...T^a is a matrix in flavour space and the color indices are summed over since a meson is a singlet of colour. The proper choice of the matrix T^a ensures the right flavour composition or isospin (or $SU(3)$) flavour quantum numbers. In order to improve the signal it is common practice to project on the zero spatial momentum states by computing the correlator

$$C_M(x_0) = \sum_{\mathbf{x}} \langle M^a(x_0, \mathbf{x}) M^a(0, \mathbf{0})\rangle. \tag{6.45}$$

As usual the Grassmann integrations can be readily performed and the result is

$$\begin{aligned} C_M(x_0) = \frac{1}{Z[0]} \int DU e^{-S_g[U]} \det(D_W + M) \sum_{\mathbf{x}} \\ \big\{-\mathrm{Tr}[(D_W + M)^{-1}_{0,x}(\Gamma \otimes T^a)(D_W + M)^{-1}_{x,0}(\Gamma \otimes T^b)] \\ + \mathrm{Tr}[(D_W + M)^{-1}_{0,0}(\Gamma \otimes T^a)]\mathrm{Tr}[(D_W + M)^{-1}_{x,x}(\Gamma \otimes T^b)]\big\}\,. \end{aligned} \tag{6.46}$$

The two terms correspond to the connected and disconnected contributions, shown in Fig. 6.2. The latter are much harder to compute numerically because the sum over $\mathbf{x}$ would require the inversion of the Dirac operator as many times as there are spatial points, while the connected contribution can be obtained with a single inversion per spin and colour.

6.2.2 Baryons

Baryons are qqq color singlets. We can take the following operators:

$$B^{abc}_{\alpha\beta\gamma} = \psi(x)_\alpha \equiv \epsilon_{c_1c_2c_3}\psi_{\alpha ac_1}\psi_{\beta bc_2}\psi_{\gamma cc_3}, \tag{6.48}$$

where a, b, c are the flavour indices and α, β, γ the spinor ones. The contraction of this three-quark object with appropriate tensors of both set of indices will ensure the right flavour and spin respectively.

For example, consider the proton, which is a $J = 1/2$, $P = +1$ and $I = 1/2$ state made up of two u quarks and one d quark. In order to combine these three, we can first combine the d and one u in a $J = 0$, $I = 0$ diquark state and then add the third one. We need therefore to combine the u and d antisymmetrically both in flavour and spin, obtaining

$$(u_\alpha d_\beta - d_\alpha u_\beta)(C\gamma_5)_{\alpha\beta}, \tag{6.49}$$

where $C\gamma_5$ ensures that the quark states with up and down spin are combined antisymmetrically and are therefore a singlet under rotations. Since $C\gamma_5$ is antisimmetric the two terms are the same and the possible proton operator is given by

$$p_\gamma = u_\gamma u_\alpha d_\beta (C\gamma_5)_{\alpha\beta} = u^T C\gamma_5 d u_\gamma, \tag{6.50}$$

where the color indices are not shown but are contracted with the ϵ tensor.

The corresponding anti-proton is

$$\bar{p}_\gamma = \bar{d} C\gamma_5 \bar{u}^T \bar{u}_\gamma. \tag{6.51}$$

The two-point correlation functions of those operators at large x_0 separation, are dominated by the lightest one-particle state in the corresponding channel:

$$\lim_{x_0 \to 0} \sum_{\mathbf{x}} \langle B(x) B(0) \rangle = \lim_{x_0 \to 0} \sum_{\mathbf{x}} \langle 0 | T(\hat{B}(x) \hat{B}(0)) | 0 \rangle_E = \frac{Z_L}{2} (1 + \gamma_0) \, e^{-m_L x_0}, \tag{6.52}$$

where m_L is the mass of the lightest state in this channel, $|L\rangle$, and $Z_L = |\langle 0 | \hat{B}(0) | L \rangle|^2$, the vacuum-to-this-state matrix element.

Exercise 5.3 Write down an interpolating operator for the Ω baryon ($J^P = 3/2^+$) made of three strange quarks and an interpolating operator for the ρ^+ meson.

6.3 Decay constants: pion to vacuum matrix elements

A consequence of the chiral Ward indentity is the PCAC relation, ie. the coupling of the axial current to the single pseudoscalar meson states, the lightest of them being the pion $|\pi\rangle$. The corresponding matrix element is the decay constant. This is the matrix element needed for determining the leptonic decays widths of pseudoscalar mesons, from which several of the elements of the CKM matrix are best determined.

$$\langle 0 | A_\mu^a(x) | \pi(p) \rangle = i F_\pi p_\mu e^{-ipx}. \tag{6.53}$$

Therefore F_π can be determined from the normalization of the axial-current two-point correlator provided it is appropriately renormalized:

$$-\lim_{x_0\to\infty} Z_A^2 \sum_{\mathbf{x}} \langle A_0(x)A_0(0)\rangle = \frac{F_\pi^2 M_\pi}{2}\exp(-M_\pi x_0). \tag{6.54}$$

An essential requirement is therefore to obtain Z_A. I refer to P. Weisz's (**?**) and A. Vladikas's lectures (Vladikas, 2009).

6.4 Form factors: single state matrix elements of current operators

In order to describe other processes, such as meson semileptonic decays in which a meson decays into a lighter one emiting two leptons, e.g. $B \to \pi l \nu_l$ (important in the determination of V_{ub}) we need to know the matrix element of the weak current between the two initial and final meson states.

$$\langle M|\bar{q}T^a\gamma_\mu(1-\gamma_5)q|M'\rangle, \tag{6.55}$$

where the flavour quantum numbers of M, M' and T^a should be appropriately fixed for the given process. According to the LSZ reduction formulae, this matrix element can be obtained from the expectation value of the time ordered product of three operators: the vector current and the two operators that have an overlap with the initial and final meson states,

$$\lim_{x_0,y_0\to+\infty,-\infty} \sum_{\mathbf{x},\mathbf{y}} \langle M^a(x)J_\mu^b(0)M^c(y)\rangle. \tag{6.56}$$

In contrast with two point functions that depend on a single momentum, the three-point functions depend on two and therefore the matrix element has a non-trivial momentum dependence dictated by Lorentz invariance such as:

$$\langle \pi(p)|J_\mu(q)|B(p')\rangle = f^+(q^2)\left[p'+p-\frac{m_B^2-m_\pi^2}{q^2}q\right]_\mu + f^0(q^2)\frac{m_B^2-m_\pi^2}{q^2}q_\mu. \tag{6.57}$$

The coefficients $f^+(q^2)$, $f^0(q^2)$ are called form factors and in principle they must be determined in the whole kinematical range of q^2.

6.5 Two-body decays

Other processes such as $K \to \pi\pi, \rho \to \pi\pi$, etc involve also three-point functions. However their large time behaviour does not contain sufficient information to reconstruct the corresponding S-matrix element (Maiani and Testa, 1990). A similar problem affects other scattering processes.

It is important to point out that there is nothing wrong with LSZ reduction formula on the Euclidean infinite lattice(Lüscher, 1988). Any S-matrix element can be computed by:

- computing the connected Euclidean correlation functions in momentum space

$$\sum_{x_n}...\sum_{x_1} e^{-iq_1x_1}...e^{-iq_nx_n}\langle O(x_1)....O(x_n)\rangle = S_n(q_1,...,q_n), \tag{6.58}$$

- Wick rotating them back to Minkowski:

$$W_n(E_1, ...E_n) = S_n(q_1, ..., q_n)|_{q_i^0=(i-\epsilon)E_i} \, . \tag{6.59}$$

- The S-matrix element is the given by

$$\langle \mathbf{p}_3, ..., \mathbf{p}_n; out|\mathbf{p_1}\mathbf{p}_2\rangle = \prod_k \frac{(E_k^2 - \omega(\mathbf{p}_k))}{\sqrt{Z_k}} W_n \Bigg|_{E_i=\pm\omega(\mathbf{p}_i)} \, . \tag{6.60}$$

This method is however numerically hopeless. There are smarter ways to go around, by using finite-size scaling techniques. QCD in a box is a wonderfull laboratory from which physical information can be extracted. A few examples of the uses of a finite volume are

- Finite-size dependence of one particle masses is related to the forward elastic scattering amplitude (Lüscher, 1983; Lüscher, 1986*a*)
- Two particle spectra in a box is related to the scattering phase shifts and unstable particle widths(Lüscher, 1986*b*). See the lectures of S. Aoki (Aoki, 2009) were some applications are discussed.
- The Nambu-Goldstone bosons in a box behave in a way that can be predicted by Chiral Perturbation Theory and provides a different regime to match QCD with the chiral Lagrangian: the so-called ϵ-regime (Gasser and Leutwyler, 1987)
- Non-perturbative renormalization: the renormalization scale is set by the box size (Jansen *et al.*, 1996). See the lectures of (Weisz, 2009).

and the list is probably not exhausted...

An important message is that in lattice QCD simulations the optimal conditions to extract physical parameters are not necessarily the same conditions as in real experiments. We surely need to prove the universality of our results by taking the limit $a \to 0$, but we should also exploit as much as possible the possibilities that the lattice offers of probing QCD in new conditions (unphysical quark masses, finite volume, etc...)

Acknowledgements

I wish to thank the organizers of this wonderful school: L. Lellouch, R. Sommer, B. Svetitsky and especially A. Vladikas for their support and their invaluable help in editing these proceedings. I profited greatly from the critical reading of these lectures by F. Bernardoni and A. Donini. I would also like to thank my co-lecturers S. Aoki, D. Kaplan, M. Golterman and P. Weisz for their advise, their patient sitting through my lectures, and even answering some hard questions for me. Last but not least I wish to warmly thank the students for the stimulating enviroment of the school, and the director of the school Leticia Cugliandolo for her support in academic as well as family matters.

References

Adler, S. L. and Bardeen, W. A. (1969). Absence of higher order corrections in the anomalous axial vector divergence equation. *Phys. Rev.*, **182**, 1517–1536.

Amsler, C. et al. (2008). Review of particle physics. *Phys. Lett.*, **B667**, 1.

Aoki, S. (2009). Lectures on Lattice QCD and Nuclear Physics, Les Houches École d'Été de Physique Théorique.

Appelquist, T. and Carazzone, J. (1975). Infrared Singularities and Massive Fields. *Phys. Rev.*, **D11**, 2856.

Atiyah, M. F. and Singer, I. M. (1971). The Index of elliptic operators. 5. *Annals Math.*, **93**, 139–149.

Belavin, A. A., Polyakov, A. M., Schwartz, A. S., and Tyupkin, Y. S. (1975). Pseudoparticle solutions of the Yang-Mills equations. *Phys. Lett.*, **B59**, 85–87.

Berezin, F. A. (1966). *The method of second quantization.* Academic Press, New York.

Bochicchio et al. (1985). Chiral Symmetry on the Lattice with Wilson Fermions. *Nucl. Phys.*, **B262**, 331.

Bogoliubov, N. and Parasiuk, O. (1957). On the Multiplication of the causal function in the quantum theory of fields. *Acta Math.*, **97**, 227–266.

Callaway, David J. E. (1988). Triviality Pursuit: Can Elementary Scalar Particles Exist? *Phys. Rept.*, **167**, 241.

Collins, J. C. (1984). *Renormalization.* Cambridge University Press.

Creutz, M. (1983). *Quarks, gluons and lattices* . Cambridge Monographs on Mathematical Physics, Cambridge University Press.

DeGrand, T. and DeTar, C. (2006). *Lattice Methods for Quantum Chromodynamics* . World Scientific Publishing.

Feynman, R. (1948). The Space-Time Formulation of Nonrelativistic Quantum Mechanics. *Rev. Mod. Phys.*, **20**, 367.

Gasser, J. and Leutwyler, H. (1987). Thermodynamics of Chiral Symmetry. *Phys. Lett.*, **B188**, 477.

Gattringer, C. and Lang, C.B. (2010). *Quantum Chromodynamics on the Lattice: An Introductory Presentation.* Lect. Notes Phys. 788, Springer Berlin Heidelberg.

Goldstone, J. (1961). Field Theories with Superconductor Solutions. *Nuovo Cim.*, **19**, 154–164.

Golterman, M. (2009). Lectures on Applications of chiral perturbation theory to lattice QCD, Les Houches École d'Été de Physique Théorique.

Hasenfratz, A. et al. (1987). The Triviality Bound of the Four Component phi**4 Model. *Phys. Lett.*, **B199**, 531.

Hasenfratz, P., Laliena, V., and Niedermayer, F. (1998). The index theorem in QCD with a finite cut-off. *Phys. Lett.*, **B427**, 125–131.

Hepp, K. (1966). Proof of the Bogolyubov-Parasiuk theorem on renormalization. *Commun. Math. Phys.*, **2**, 301–326.

Itzykson, C. and Zuber, J. B. (1980). *Quantum Field Theory.* McGraw-Hill Inc.

Jansen, K. et al. (1996). Non-perturbative renormalization of lattice QCD at all scales. *Phys. Lett.*, **B372**, 275–282.

Källen, G. (1972). Quantum Electrodynamics. Springer-verl./new York 1972, 233p.

Kaplan, D. B. (2009). Lectures on Chiral Symmetry and Lattice Fermions, Les Houches École de Physique.

Kogut, J. B. and Susskind, L. (1975). Hamiltonian Formulation of Wilson's Lattice Gauge Theories. *Phys. Rev.*, **D11**, 395.

Lehmann, H (1954). On the Properties of propagation functions and renormalization contants of quantized fields. *Nuovo Cim.*, **11**, 342–357.

Lehmann, H., Symanzik, K., and Zimmermann, W. (1955). On the formulation of quantized field theories. *Nuovo Cim.*, **1**, 205–225.

Lüscher, M. (1977). Construction of a Selfadjoint, Strictly Positive Transfer Matrix for Euclidean Lattice Gauge Theories. *Commun. Math. Phys.*, **54**, 283.

Lüscher, Martin (1982). Topology of Lattice Gauge Fields. *Commun. Math. Phys.*, **85**, 39.

Lüscher, M. (1983). On a Relation between finite size effects and elastic scattering processes. Lecture given at Cargese Summer Inst., Cargese, France, Sep 1-15, 1983.

Lüscher, M. (1986*a*). Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States. *Commun. Math. Phys.*, **104**, 177.

Lüscher, M. (1986*b*). Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States. *Commun. Math. Phys.*, **105**, 153–188.

Lüscher, M. (1988). Selected topics in lattice field theory. Lectures given at Summer School 'Fields, Strings and Critical Phenomena', Les Houches, France, Jun 28 - Aug 5, 1988.

Lüscher, M. (2009). Lectures on Computational Strategies, Les Houches École d'Été de Physique Théorique.

Lüscher, M. and Weisz, P. (1988). Is There a Strong Interaction Sector in the Standard Lattice Higgs Model? *Phys. Lett.*, **B212**, 472.

Lüscher, M. and Weisz, P. (1989). Scaling Laws and Triviality Bounds in the Lattice phi**4 Theory. 3. N Component Model. *Nucl. Phys.*, **B318**, 705.

Maiani, L. and Testa, M. (1990). Final state interactions from Euclidean correlation functions. *Phys. Lett.*, **B245**, 585–590.

Menotti, P. and Pelissetto, A. (1987). General proof of Osterwalder-Schrader positivity for the Wilson Action. *Commun. Math. Phys.*, **113**, 369.

Montvay, I. and Münster, G. (1994). *Quantum Fields on a Lattice.* Cambridge Monographs on Mathematical Physics, Cambridge University Press.

Montvay, I., Münster, G., and Wolff, U. (1988). Percolation cluster algorithm and scaling behaviour in the four-dimensional Ising model. *Nucl. Phys.*, **B305**, 143.

Nambu, Y. (1960). Quasi-particles and gauge invariance in the theory of superconductivity. *Phys. Rev.*, **117**, 648–663.

Nielsen, H. B. and Ninomiya, M. (1981). No Go Theorem for Regularizing Chiral Fermions. *Phys. Lett.*, **B105**, 219.

Osterwalder, K. and Schrader, R. (1973). Axioms for Euclidean Green's Functions. *Commun. Math. Phys.*, **31**, 83–112.

Osterwalder, K. and Schrader, R. (1975). Axioms for Euclidean Green's Functions. 2. *Commun. Math. Phys.*, **42**, 281.

Osterwalder, K. and Seiler, E. (1978). Gauge Field Theories on the Lattice. *Ann. Phys.*, **110**, 440.

Peskin, M. E. and Schroeder, D. V. (1995). *An Introduction to Quantum Field Theory* . Addison-Wesley Publishing Company.

Philipsen, U. (2009). Lectures on QCD at Finite Temperature and Quark Density, Les Houches École d'Été de Physique Théorique.

Reisz, T. (1988). A power counting theorem for Feynman integrals on the lattice. *Commun. Math. Phys.*, **116**, 81.

Rothe, H.J. (2005). *Lattice Gauge Theoris: An Introduction* . World Scientific Lecture Notes in Physics- Vol. 74.

Smit, J. (1991). Transfer operator and lattice fermions. *Nucl. Phys. Proc. Suppl.*, **20**, 542–545.

Smit, J. (2002). *Introduction to Quantum Fields on a Lattice.* Cambridge Lecture Notes in Physics, Cambridge University Press.

Sommer, R. (2009). Lectures on Heavy Quark Effective Theory, Les Houches École d'Été de Physique Théorique.

Streater, R. and Wightman, A. S. (1964). *PCT, Spin and Statistics and All That* . Mathematical Physics Monograph Series, Benjamin, New York.

Susskind, L. (1977). Lattice Fermions. *Phys. Rev.*, **D16**, 3031–3039.

't Hooft, G. (1976). Symmetry breaking through Bell-Jackiw anomalies. *Phys. Rev. Lett.*, **37**, 8–11.

Takahashi, Y. (1957). On the generalized Ward identity. *Nuovo Cim.*, **6**, 371.

Veneziano, G. (1979). U(1) Without Instantons. *Nucl. Phys.*, **B159**, 213–224.

Vladikas, A. (2009). Lectures on Non-perturbative Renormalization, Les Houches École d'Été de Physique Théorique.

Ward, J. C. (1950). An Identity in Quantum Electrodynamics. *Phys. Rev.*, **78**, 182.

Weinberg, S. (1995). *The Quantum Theory of Fields, Vol. 1: Foundations* . Cambridge University Press.

Weisz, P. (2009). Lectures on Renormalization and Improvement, Les Houches École d'Été de Physique Théorique.

Wilson, K.G. (1974). Confinement of quarks. *Phys. Rev.*, **D10**, 2445–2459.

Wilson, K. G. (1975). Quarks and Strings on a Lattice. New Phenomena In Subnuclear Physics. Part A. Proceedings of the First Half of the 1975 International School of Subnuclear Physics, Erice, Sicily, July 11 - August 1, 1975, ed. A. Zichichi, Plenum Press, New York, 1977, p. 69, CLNS-321.

Wilson, K. G. and Kogut, J. B. (1974). The Renormalization group and the epsilon expansion. *Phys. Rept.*, **12**, 75–200.

Witten, E. (1979). Current Algebra Theorems for the U(1) Goldstone Boson. *Nucl. Phys.*, **B156**, 269.

Zimmermann, W. (1969). Convergence of Bogolyubov's method of renormalization in momentum space. *Commun. Math. Phys.*, **15**, 208–234.