Leaving diagrams behind: Anomalies through functional methods

Anomalies and Jacobians

Emmy Noether
(1882-1935)

Classically, continuous symmetries lead to conserved currents through Noether's theorem.

Take a theory with action $S[\phi_i]$ invariant under **global** transformations

$$\delta_\xi \phi_i(x) = \sum_j \xi_j F_{ij}(\phi_k)$$

The conserved current can be obtained using "Noether's trick". Taking $\xi_i(x)$ to depend on the position

$$\begin{aligned} S[\phi_i + \delta_\epsilon \phi_i] &= S[\phi_i] - \sum_i \int d^4x\, \partial_\mu \xi_i(x) j_i^\mu(x) \\ &= S[\phi_i] + \sum_i \int d^4x\, \xi_i(x) \partial_\mu j_i^\mu(x) \end{aligned}$$

If the fields are on-shell, the action is invariant under any variation $\xi_i(x)$

$$\sum_i \int d^4x\, \xi_i(x) \partial_\mu j_i^\mu(x) = 0 \quad \longrightarrow \quad \partial_\mu j_i^\mu(x) = 0$$

Let us move to the **quantum theory**. We look at a generic correlation function

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

We apply now a **change of variables** inside the integral

$$\phi_i'(x) = \phi_i(x) + \delta_\xi\phi_i(x)$$

that does not change its value

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1'(x_1)\ldots\mathscr{O}_n'(x_n)e^{\frac{i}{\hbar}S[\phi_i']}$$

where $\mathscr{O}_i'(x)$ is the transformation of the operator $\mathscr{O}_i(x)$. At **first order**

$$\mathscr{O}_i'(x) = \mathscr{O}_i(x) + \delta_\xi\mathscr{O}_i(x)$$

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1'(x_1)\ldots\mathscr{O}_n'(x_n)e^{\frac{i}{\hbar}S[\phi_i']}$$

$$\mathscr{O}_i'(x) = \mathscr{O}_i(x) + \delta_\xi\mathscr{O}_i(x)$$

$$S[\phi_i'] = S[\phi_i] + \sum_i\int d^4x\,\xi_i(x)\partial_\mu j_i^\mu(x)$$

Combining these identities and expanding to **linear order** in ξ_i

$$\begin{aligned}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle &= \frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]} \\ &+ \frac{i}{\hbar}\frac{1}{Z}\sum_k\int d^4x\,\xi_k(x)\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\,e^{\frac{i}{\hbar}S[\phi_i]} \\ &+ \frac{1}{Z}\sum_{a=1}^n\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}\end{aligned}$$

$$\begin{aligned}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle &= \frac{1}{Z}\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}\\ &+ \frac{i}{\hbar}\frac{1}{Z}\sum_k\int d^4x\,\xi_k(x)\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\,e^{\frac{i}{\hbar}S[\phi_i]}\\ &+ \frac{1}{Z}\sum_{a=1}^n\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}\end{aligned}$$

Now we make a further assumption

$$\prod_i\mathscr{D}\phi_i' = \prod_i\mathscr{D}\phi_i$$

and arrive at the **Ward identity**:

$$\begin{aligned}\frac{i}{\hbar}\sum_k\int d^4k\,\xi_k(x)\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\Big]|\Omega\rangle\\ = \sum_{a=1}^n\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle\end{aligned}$$

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

$$+\ \frac{i}{\hbar}\frac{1}{Z}\sum_k\int d^4x\,\xi_k(x)\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\,e^{\frac{i}{\hbar}S[\phi_i]}$$

$$+\ \frac{1}{Z}\sum_{a=1}^n\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

However, we can also have a **nontrivial Jacobian** in the functional integral

$$\prod_i \mathscr{D}\phi_i' = \left[1+\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\right]\prod_i\mathscr{D}\phi_i$$

This introduces an extra term in the Ward identity

$$\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\int\left[\prod_i\mathscr{D}\phi_i\right]\mathscr{O}(x_1)\ldots\mathscr{O}(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

$$+\frac{i}{\hbar}\frac{1}{Z}\sum_k\int d^4x\,\xi_k(x)\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\,e^{\frac{i}{\hbar}S[\phi_i]}$$

$$+\frac{1}{Z}\sum_{a=1}^{n}\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

However, we can also have a **nontrivial Jacobian** in the functional integral

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$$\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\int\left[\prod_i\mathscr{D}\phi_i\right]\mathscr{O}(x_1)\ldots\mathscr{O}(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

This gives the **anomalous Ward identity**.

$$
\begin{aligned}
-\frac{i}{\hbar}\sum_k \int d^4k\,\xi_k(x)\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j^\mu_k(x)\Big]|\Omega\rangle \\
= \sum_{a=1}^{n}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle \\
+ \left[\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\right]\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle
\end{aligned}
$$

For the particular case in which $\mathscr{O}_i(x)\equiv\mathbb{I}$

$$
\sum_k\int d^4x\,\xi_k(x)\langle\Omega|\partial_\mu j^\mu_k(x)|\Omega\rangle = i\hbar\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)
$$

$$\Downarrow\quad \forall\ \xi_k(x)$$

$$
\langle\Omega|\partial_\mu j^\mu_k(x)|\Omega\rangle = i\hbar\,\mathscr{J}_k(x)
$$

The anomaly is given by the functional Jacobian!

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$$
\begin{aligned}
-\frac{i}{\hbar}\sum_k \int d^4k\, \xi_k(x)\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\Big]|\Omega\rangle
&= \sum_{a=1}^{n}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle \\
&+ \left[\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\right]\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle
\end{aligned}
$$

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$$

The anomaly is given by the functional Jacobian!

The fermion effective action

Foreword: Euclidean fermion fields

In Minkowski space, the Dirac matrices satisfy $[\,\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)]$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \quad \Longrightarrow \quad \begin{cases} \gamma^{0\dagger} = \gamma^0 \\ \gamma^{i\dagger} = -\gamma^i \end{cases}$$

Dirac fermions are defined as objects transforming under the Lorentz group as

$$\psi' = e^{-\frac{i}{2}\vartheta_{\mu\nu}\sigma^{\mu\nu}}\psi \equiv U(\vartheta)\psi \qquad \text{where} \qquad \begin{cases} \sigma^{\mu\nu} = -\dfrac{i}{4}[\gamma^\mu,\gamma^\nu] \\ \sigma^{0i\dagger} = -\sigma^{0i}, \qquad \sigma^{ij\dagger} = \sigma^{ij}. \end{cases}$$

Since $\sigma^{\mu\nu}$ is not Hermitian, Hermitian conjugate spinors are not "contravariant"

$$\psi^{\dagger\prime} = \psi^\dagger e^{\frac{i}{2}\vartheta_{\mu\nu}\sigma^{\mu\nu\dagger}} \equiv \psi^\dagger U(\vartheta)^\dagger \neq \psi^\dagger U(\vartheta)^{-1}$$

$$\sigma^{\mu\nu\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0 \quad \Downarrow \quad \gamma^0 U(\vartheta)^\dagger\gamma^0 = U(\vartheta)^{-1}$$

$$\overline{\psi}' = \psi^{\dagger\prime}\gamma^0 = \psi^\dagger U(\vartheta)^\dagger\gamma^0 = \psi^\dagger\gamma^0 U(\vartheta)^{-1} = \overline{\psi}U(\vartheta)^{-1}$$

Euclidean space can be obtained by Wick rotation from Minkowski signature

$$x^0 = -ix^4 \quad \Longrightarrow \quad \eta_{\mu\nu} \longrightarrow -\delta_{\mu\nu}$$

while the new Dirac matrices are defined as

$$\left.\begin{array}{l} \widehat{\gamma}^4 = i\gamma^0 \\ \widehat{\gamma}^i = \gamma^i \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} \{\widehat{\gamma}^\mu, \widehat{\gamma}^\nu\} = -2\delta^{\mu\nu}\mathbb{I} \\ \widehat{\gamma}^{\mu\dagger} = -\widehat{\gamma}^\mu \end{array}\right.$$

Euclidean Dirac fermions are objects transforming under $\mathrm{SO}(4)$ as

$$\psi' = e^{-\frac{i}{2}\omega_{\mu\nu}\widehat{\sigma}^{\mu\nu}}\psi \equiv O(\omega)\psi \quad \Longrightarrow \quad \left\{\begin{array}{l} \widehat{\sigma}^{\mu\nu} = \dfrac{i}{4}[\widehat{\gamma}^\mu, \widehat{\gamma}^\nu] \\ \widehat{\sigma}^{\mu\nu\dagger} = \widehat{\sigma}^{\mu\nu} \end{array}\right.$$

Now, Hermitian conjugate objects are **contravariant**

$$\psi'^\dagger = \psi^\dagger e^{\frac{i}{2}\omega_{\mu\nu}\widehat{\sigma}^{\mu\nu\dagger}} \equiv \psi^\dagger O(\omega)^\dagger = \psi^\dagger O(\omega)^{-1}$$

In Euclidean space, the chirality matrix is defined as

$$\widehat{\gamma}_5 = -\widehat{\gamma}^1\widehat{\gamma}^2\widehat{\gamma}^3\widehat{\gamma}^4$$

satisfying

$$\widehat{\gamma}_5^\dagger = \widehat{\gamma}_5$$

A particularly important identity in the computation of anomalies is

$$\mathrm{Tr}\left(\widehat{\gamma}_5\widehat{\gamma}^\mu\widehat{\gamma}^\nu\widehat{\gamma}^\alpha\widehat{\gamma}^\beta\right) = -4\epsilon^{\mu\nu\alpha\beta} \qquad \text{where} \qquad \epsilon^{1234} = 1$$

Comparing with its Minkowskian counterpart

$$\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = -4i\epsilon^{\mu\nu\alpha\beta} \qquad \text{with} \qquad \epsilon^{0123} = 1$$

we see how Euclidean chiral anomalies will have an **addition** factor of i.

Then, the Euclidean action for a Dirac fermion is

$$S_E = \int d^4x\, \psi^\dagger \Big(i\widehat{\gamma}^\mu \partial_\mu - m\Big)\psi$$

which leads to the propagator

$$\langle 0|T\Big[\psi_\alpha(x)\psi^\dagger_\beta(y)\Big]|0\rangle = \int \frac{d^4p}{(2\pi)^4}\left(\frac{1}{p_\mu\widehat{\gamma}^\mu - m}\right)_{\alpha\beta} e^{-ip\cdot(x-y)}$$

This equation, however, is not homogeneous under Hermitian conjugation!

The way out to this problem is to take the Euclidean Dirac action

$$S_E = \int d^4x\, \overline{\psi}\Big(i\widehat{\gamma}^\mu \partial_\mu - m\Big)\psi$$

with $\psi(x)$ and $\overline{\psi}(x)$ as **independent** fields.

Thus, in Euclidean space $\overline{\psi}(x)$ transforms **contravariantly** and

$\overline{\psi}(x) \neq \psi(x)\widehat{\gamma}^0$ (despite the misleading notation)

Remember also that the representations of the Lorentz group $\mathrm{SO}(1{,}3)$ can be written as the **product of two copies** of $\mathrm{SU}(2)$

$$\mathcal{J}_k^{\pm} = \frac{1}{2}(J_k \pm iK_k) \qquad \left\{ \begin{array}{ll} J_k^{\dagger} = J_k & \text{rotations} \\ \\ K_k^{\dagger} = K_k & \text{boosts} \end{array} \right.$$

These generators satisfy

$$[\mathcal{J}_k^{\pm}, \mathcal{J}_\ell^{\pm}] = i\epsilon_{k\ell j}\mathcal{J}_j^{\pm} \qquad\qquad [\mathcal{J}_k^{\pm}, \mathcal{J}_\ell^{\mp}] = 0$$

Thus, any representation of the Lorentz group can be written as a representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ labelled by

$$(\mathbf{s}_+, \mathbf{s}_-)$$

Since $\mathcal{J}_k^{\pm\dagger} = \mathcal{J}_k^{\mp}$ Hermitian conjugation interchanges the labels. In particular

$$(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) \longrightarrow (\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$$

In the case of $\mathrm{SO}(4)$, its representations can also be written in terms of those of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ using the **'t Hooft symbols**:

$$\eta^a_{\mu\nu} = \varepsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{\mu 4}$$

$$\overline{\eta}^a_{\mu\nu} = \varepsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 4} + \delta_{a\nu}\delta_{\mu 4}$$

where $\varepsilon_{a\mu\nu}$ is the 3D antisymmetric symbol with $\varepsilon_{a\mu\nu} = 0$ whenever μ or ν take the value 4

The generators

$$N^a = \eta^a_{\mu\nu} J^{\mu\nu} \qquad \overline{N}^a = \overline{\eta}^a_{\mu\nu} J^{\mu\nu}$$

satisfy the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ algebra

$$[N^a, N^b] = i\varepsilon^{abc} N^c \qquad [\overline{N}^a, \overline{N}^b] = i\varepsilon^{abc}\overline{N}^c \qquad [N^a, \overline{N}^b] = 0$$

while N^a and $\overline{N}^a$ are **not related** by Hermitian conjugation.

Notation **WARNING**

From now on, Euclidean gamma matrices will be "**hatless**"

The fermion effective action

From now on we work in **Euclidean space**.

In the computation of anomalies, it is convenient to work with the **one-loop fermion effective action**. In the case of QED, this is

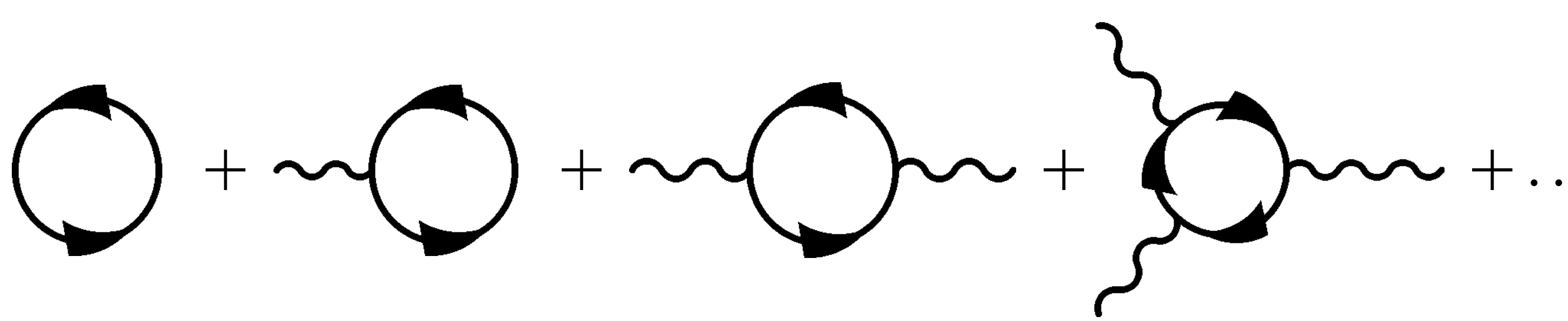
$$e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \exp\left[-\int d^4y\,\overline{\psi}\Big(i\partial\!\!\!/ + e\mathscr{A}\!\!\!/\Big)\psi\right]$$

This effective action is a **nonlocal** functional.

we are integrating out a massless fermion

Expanding the integrand in powers of the electric charge e, the effective action can be written as the sum of **one-loop** diagrams:

$\Gamma[\mathscr{A}] =$



Consider a massless Dirac fermion coupled to **external** axial and vector Abelian gauge fields

$$S=\int d^4x\,\overline{\psi}\Big(i\partial\!\!\!/+\mathcal{V}\!\!\!/+\mathcal{A}\!\!\!/\gamma_5\Big)\psi$$

This theory has two types of **local** invariances:

Vector	**Axial-vector**
$\psi(x)\longrightarrow e^{i\alpha(x)}\psi(x)$	$\psi(x)\longrightarrow e^{i\beta(x)\gamma_5}\psi(x)$
$\overline{\psi}(x)\longrightarrow\overline{\psi}(x)e^{-i\alpha(x)}$	$\overline{\psi}(x)\longrightarrow\overline{\psi}(x)e^{i\beta(x)\gamma_5}$
$\mathcal{V}_\mu(x)\longrightarrow\mathcal{V}_\mu(x)+\partial_\mu\alpha(x)$	$\mathcal{V}_\mu(x)\longrightarrow\mathcal{V}_\mu(x)$
$\mathcal{A}_\mu(x)\longrightarrow\mathcal{A}_\mu(x)$	$\mathcal{A}_\mu(x)\longrightarrow\mathcal{A}_\mu(x)+\partial_\mu\beta(x)$
$J^\mu_{\rm V}=\overline{\psi}\gamma^\mu\psi$	$J^\mu_{\rm A}=\overline{\psi}\gamma^\mu\gamma_5\psi$

For this theory, the **fermion effective action** is defined as

$$e^{-\Gamma[\mathcal{V},\mathcal{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \; \exp\left[-\int d^4y\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right]$$

To find why this definition is useful, let us take the functional derivative

$$\begin{aligned}\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]} &= -\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\,\overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)\exp\left[-\int d^4y\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right] \\ &= -\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\,J^\mu_{\mathrm{A}}(x)\exp\left[-\int d^4y\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right]\end{aligned}$$

while the left-hand side can be written as

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]} = -e^{-\Gamma[\mathcal{V},\mathcal{A}]}\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}]$$

$$\Downarrow$$

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = -e^{\Gamma[\mathcal{V},\mathcal{A}]}\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]}$$

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]} = -\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, J^\mu_{\rm A}(x)\exp\left[-\int d^4x\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right]$$

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = -e^{\Gamma[\mathcal{V},\mathcal{A}]}\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]}$$

Combining these two identities, we arrive at

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = \langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

Moreover, the variation of the effective axion under axial-vector transformations are

$$\delta_\beta\Gamma[\mathcal{V},\mathcal{A}] = \int d^4x\,\delta_\beta\mathcal{A}_\mu(x)\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = \int d^4x\,\partial_\mu\beta(x)\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}]$$

and integrating by parts

$$\delta_\beta\Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\,\beta(x)\partial_\mu\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

Thus, the (integrated) **anomaly of the axial current** is given by the **variation** of the effective action under **axial-vector transformations**

$$\delta_\beta \Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\, \beta(x)\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

Similarly, we can compute the variation of the effective action under vector gauge transformations

$$\frac{\delta}{\delta \mathcal{V}_\mu(x)} e^{-\Gamma[\mathcal{V},\mathcal{A}]} = -\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, J^\mu_{\rm V}(x) \exp\left[-\int d^4y\, \overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right]$$

Proceeding as with the axial-vector current, we arrive at

$$\delta_\alpha \Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\, \alpha(x)\partial_\mu \langle J^\mu_{\rm V}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

Thus, the **anomaly of the vector current** is given by the **variation** of the fermion effective action under **vector gauge transformations**.

This expression of the anomaly can be connected with the existence of a **nontrivial Jacobian**. (**Fujikawa's method**)

Let us consider, for example, an axial-vector gauge transformation

$$\mathcal{V}'_\mu(x) = \mathcal{V}_\mu(x) \qquad\qquad \mathcal{A}'_\mu(x) = \mathcal{A}_\mu(x) + \partial_\mu \beta(x)$$

The transformed effective action is

$$e^{-\Gamma[\mathcal{V},\mathcal{A}']} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[\int d^4x \, \overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/'\gamma_5\Big)\psi\right]$$

However, this change in the action can be "undone" by a change of variables in the functional integral

$$\psi'(x) = e^{-i\beta(x)\gamma_5}\psi(x) \qquad\qquad \overline{\psi}'(x) = \overline{\psi}(x)e^{-i\beta(x)\gamma_5}$$

such that

$$\int d^4x \, \overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/'\gamma_5\Big)\psi = \int d^4x \, \overline{\psi}'\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi'$$

The problem arises because of the existence of a Jacobian

$$
\begin{aligned}
e^{-\Gamma[\mathcal{V},\mathcal{A}']} &= \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[\int d^4x\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/'\gamma_5\Big)\psi\right] \\
&= \int \mathscr{D}\overline{\psi}'\mathscr{D}\psi' \, \mathcal{J}[\beta] \exp\left[\int d^4x\,\overline{\psi}'\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi'\right]
\end{aligned}
$$

Now, the Jacobian is a field-independent c-number that can be taken outside the integral

$$
e^{-\Gamma[\mathcal{V},\mathcal{A}']} = \mathcal{J}[\beta] e^{-\Gamma[\mathcal{V},\mathcal{A}]} \qquad \Longrightarrow \qquad \Gamma[\mathcal{V},\mathcal{A}'] - \Gamma[\mathcal{V},\mathcal{A}] = -\log \mathcal{J}[\beta]
$$

Considering now infinitesimal axial-vector gauge transformations

Kazuo Fujikawa
(b. 1942)

$$
\delta_\beta \Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\,\beta(x)\left(\frac{1}{\mathcal{J}[\beta]}\frac{\delta \mathcal{J}[\beta]}{\delta\beta(x)}\right)\bigg|_{\beta=0}
$$

$$
\mathcal{J}[0] = 1
$$

$$
\Downarrow
$$

$$
\partial_\mu \langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathcal{V},\mathcal{A}} = \frac{\delta \mathcal{J}[\beta]}{\delta\beta(x)}\bigg|_{\beta=0}
$$

The problem arises because of the existence of a Jacobian

$$e^{-\Gamma[\mathcal{V},\mathcal{A}']} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \; \exp\left[\int d^4x \, \overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/' \gamma_5\Big)\psi\right]$$

$$= \int \mathscr{D}\overline{\psi}'\mathscr{D}\psi' \; \mathcal{J}[\beta] \exp\left[\int d \ldots\right.$$

$$\delta_\beta \Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x \, \beta(x) \partial_\mu \langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

Now, the Jacobian is a field-independent c-number that can be taken outside the integral

$$e^{-\Gamma[\mathcal{V},\mathcal{A}']} = \mathcal{J}[\beta] e^{-\Gamma[\mathcal{V},\mathcal{A}]} \qquad \Gamma[\mathcal{V},\mathcal{A}'] - \Gamma[\mathcal{V},\mathcal{A}] = -\log \mathcal{J}[\beta]$$

Considering now infinitesimal axial-vector gauge transformations

$$\delta_\beta \Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x \, \beta(x) \left(\frac{1}{\mathcal{J}[\beta]} \frac{\delta \mathcal{J}[\beta]}{\delta\beta(x)}\right)\Bigg|_{\beta=0}$$

$$\mathcal{J}[0] = 1$$

$$\partial_\mu \langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathcal{V},\mathcal{A}} = \frac{\delta \mathcal{J}[\beta]}{\delta\beta(x)}\Bigg|_{\beta=0}$$


Kazuo Fujikawa
(b. 1942)

We will implement now Fujikawa's method to compute the anomaly…

Using the usual identity for Gaussian functional integrals with Grassmann fields

$$\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, e^{-\int d^4y\overline{\psi}\mathscr{O}\psi} = \det\mathscr{O}$$

the fermion effective action can be written as a **functional determinant**:

$$\begin{aligned} e^{-\Gamma[\mathcal{V},\mathcal{A}]} &= \int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, \exp\left[-\int d^4x\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right] \\ &= \det\left(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\right) \end{aligned}$$

and therefore

$$\Gamma[\mathcal{V},\mathcal{A}] = -\log\det\left[iD\!\!\!\!/(\mathcal{V}) + \mathcal{A}\!\!\!/\gamma_5\right]$$

where we have written

$$iD\!\!\!\!/(\mathcal{V}) = i\partial\!\!\!/ + \mathcal{V}\!\!\!/$$

How to compute a functional determinant (in three slides)

Let us focus on a **positive definite** differential operator $\mathscr{O}$ satisfying the eigenvalue equation ($n = 1,2,\ldots$)

$$\mathscr{O} w_n(x) = \lambda_n w_n(x) \qquad\qquad \lambda_n > 0$$

Its determinant is **formally** defined as

$$\det \mathscr{O} = \prod_{n=1}^{\infty} \lambda_n$$

In our case, we are in fact interested in computing

$$\log\det\mathscr{O} = \sum_{n=1}^{\infty} \log\lambda_n$$

Thus, we need to find a useful **representation of the logarithm...**

Let us look at the definition of the **exponential integral**

$$E_1(z) = \int_z^\infty \frac{dt}{t} e^{-t}$$

which around $z = 0$ this function admits the expansion

$$E_1(z) = -\gamma - \log z - \sum_{\ell=1}^\infty \frac{(-z)^\ell}{\ell \ell!}$$

Now, computing

$$\begin{aligned} \int_\epsilon^\infty \frac{dt}{t} e^{-xt} &= E_1(\epsilon x) \\ &= -\log x - \gamma - \log \epsilon - \sum_{\ell=1}^\infty \frac{(-\epsilon x)^\ell}{\ell \ell!} \end{aligned}$$

we arrive at

$$\lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{dt}{t} e^{-xt} = -\log x + x\text{-independent divergent constant}$$

Eventually, we will be interested in gauge variations of the determinant (this eliminates the divergent constant). Thus, we can use the following **"definition" of the logarithm**

$$\log x = -\int_{\epsilon}^{\infty} \frac{dt}{t} e^{-xt} \qquad\qquad \epsilon \longrightarrow 0^{+}$$

With this we can write

$$\begin{aligned}\log \det \mathscr{O} &= \sum_{n=1}^{\infty} \log \lambda_n \\ &= -\int_{\epsilon}^{\infty} \frac{dt}{t} \sum_{n=1}^{\infty} e^{-t\lambda_n}\end{aligned}$$

that is,

$$\log \det \mathscr{O} = -\int_{\epsilon}^{\infty} \frac{dt}{t} \mathrm{Tr}\, e^{-t\mathscr{O}}$$

Back to the fermion effective action

Remember that we wanted to compute

$$\Gamma[\mathcal{V},\mathcal{A}] = -\log\det\left[i\not{D}(\mathcal{V}) + \not{\mathcal{A}}\gamma_5\right]$$

To make the operator **positive definite**, we compute instead

$$\Gamma[\mathcal{V},\mathcal{A}] = -\frac{1}{2}\log\det\left[i\not{D}(\mathcal{V}) + \not{\mathcal{A}}\gamma_5\right]^2$$

$$= \frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\mathrm{Tr}\, e^{-t[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}$$

$$\boxed{\log\det\mathscr{O} = -\int_\epsilon^\infty \frac{dt}{t}\mathrm{Tr}\, e^{-t\mathscr{O}}}$$

Trace in the "Dirac" and "functional" sense!

Since the anomaly of the axial current is given by

$$\delta_\beta\Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

we are left with

$$\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathcal{V},\mathcal{A}} = -\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\delta_\beta\mathrm{Tr}\, e^{-t[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}$$

$$\int d^4x\, \beta(x)\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}} = -\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\delta_\beta \mathrm{Tr}\, e^{-t[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}$$

We compute then the variation of the trace

$$-\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\delta_\beta \mathrm{Tr}\, e^{-t[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}$$

$$= \int_\epsilon^\infty dt\, \mathrm{Tr}\left\{\delta_\beta\left[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\right]\left[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\right] e^{-t[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}\right\}$$

The interesting thing is that the integrand can be written now as a **total derivative**

$$-\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\delta_\beta \mathrm{Tr}\, e^{-t[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}$$

$$= -\int_\epsilon^\infty dt\,\frac{d}{dt}\mathrm{Tr}\left\{\frac{\delta_\beta\left[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\right]}{\left[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\right]} e^{-t[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}\right\}$$

$$-\frac{1}{2}\int_{\epsilon}^{\infty}\frac{dt}{t}\delta_\beta \mathrm{Tr}\, e^{-t[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2} = \mathrm{Tr}\left\{\frac{\delta_\beta\Big[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\Big]}{\Big[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\Big]}e^{-\epsilon[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}\right\}$$

We are left with the evaluation of the variation of the operator. Recalling

$$i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5 \longrightarrow e^{-i\beta(x)\gamma_5}\Big[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\Big]e^{-i\beta(x)\gamma_5}$$

and infinitesimally,

$$\delta_\beta\Big[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\Big] = \{-i\beta\gamma_5, i\not{D}(\mathcal{V})\}$$

$$\Downarrow$$

$$\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathcal{V},\mathcal{A}} = \mathrm{Tr}\left\{\frac{\{-i\beta\gamma_5, i\not{D}(\mathcal{V})\}}{\Big[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\Big]}e^{-\epsilon[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}\right\}$$

$$\int d^4x\,\beta(x)\partial_\mu\langle J_\mathrm{A}^\mu(x)\rangle_{\mathcal{V},\mathcal{A}} = \mathrm{Tr}\left\{\frac{\{-i\beta\gamma_5, i\not{D}(\mathcal{V})\}}{\left[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5\right]}e^{-\epsilon[i\not{D}(\mathcal{V})+\not{\mathcal{A}}\gamma_5]^2}\right\}$$

The introduction of the axial-vector gauge field was a **computational trick**. To recover our result for the **axial anomaly** we set

$$\mathcal{V}_\mu = e\mathscr{A}_\mu \qquad\qquad \mathcal{A}_\mu = 0$$

The integrated Euclidean axial anomaly is then given by

$$\int d^4x\,\beta(x)\partial_\mu\langle J_\mathrm{A}^\mu(x)\rangle_{\mathscr{A}} = \mathrm{Tr}\left\{\frac{\{-i\beta\gamma_5, i\not{D}(\mathscr{A})\}}{i\not{D}(\mathscr{A})}e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}$$

$$= -2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}$$

$$\int d^4x\, \beta(x)\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\mathrm{Tr}\,\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}$$

To compute the trace, we introduce a **basis** $|\phi_k\rangle$ in the space of functions

Dirac trace

$$-2i\mathrm{Tr}\,\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4k\langle\phi_k|\beta\,\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}|\phi_k\rangle$$

$$= -2i\int d^4x\int d^4x'\int d^4k\,\langle\phi_k|x\rangle\,\langle x|\beta\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}|x'\rangle\,\langle x'|\phi_k\rangle$$

Using **locality**, we write

$$\langle x|\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}|x'\rangle = \delta^{(4)}(x-x')\beta(x)\mathrm{Tr}\,\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}$$

$$-2i\mathrm{Tr}\,\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int d^4k\,\phi_k(x)^*\mathrm{Tr}\,\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}\phi_k(x)$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int d^4k\,\phi_k(x)^*\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}\phi_k(x)$$

Since we can use any complete set of functions, we choose a set of plane waves

$$\phi_k(x) = \frac{1}{(2\pi)^2}e^{ik\cdot x}$$

$$\Downarrow$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}e^{ik\cdot x}$$

Now we have to compute the trace over the Dirac indices:

$$\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}$$

$$
\begin{aligned}
[i\not{D}(\mathscr{A})]^2 &= -\gamma^\mu\gamma^\nu D_\mu(\mathscr{A})D_\nu(\mathscr{A}) \\
&= -\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}D_\mu(\mathscr{A})D_\nu(\mathscr{A}) - \frac{1}{2}\gamma^\mu\gamma^\nu[D_\mu(\mathscr{A}),D_\nu(\mathscr{A})]
\end{aligned}
$$

Using the **Euclidean** Dirac algebra $\{\gamma^\mu,\gamma^\nu\} = -2\delta^{\mu\nu}$

$$
[i\not{D}(\mathscr{A})]^2 = D(\mathscr{A})^2 - \frac{1}{2}\gamma^\mu\gamma^\nu[D_\mu(\mathscr{A}),D_\nu(\mathscr{A})]
$$

while the second term gives the **background field strength**

$$
[D_\mu(\mathscr{A}),D_\nu(\mathscr{A})] = -ie\mathscr{F}_{\mu\nu}
$$

$$
[\not{D}(\mathscr{A})]^2 = D(\mathscr{A})^2 + \frac{ie}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}
$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}e^{ik\cdot x}$$

$$\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = \mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[D(\mathscr{A})^2+\frac{i}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right\}$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[D(\mathscr{A})^2+\frac{ie}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right\}e^{ik\cdot x}$$

To further simplify, we use

$$[D_\mu(\mathscr{A}),e^{ik\cdot x}] = ik_\mu e^{ik\cdot x} \longrightarrow D_\mu(\mathscr{A})e^{ik\cdot x} = e^{ik\cdot x}[D_\mu(\mathscr{A})+ik_\mu]$$

and this leads to:

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{-\epsilon\{[D(\mathscr{A})+ik]^2+\frac{ie}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

$$k_\mu \to \frac{1}{\sqrt{\epsilon}}k_\mu$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{-\{[\sqrt{\epsilon}D(\mathscr{A})+ik]^2+\frac{ie}{2}\epsilon\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}e^{ik\cdot x}$$

$$\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = \mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[D(\mathscr{A})^2+\frac{i}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right\}$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[D(\mathscr{A})^2+\frac{ie}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right\}e^{ik\cdot x}$$

To further simplify, we use

$$[D_\mu(\mathscr{A}),e^{ik\cdot x}] = ik_\mu e^{ik\cdot x} \quad\longrightarrow\quad D_\mu(\mathscr{A})e^{ik\cdot x} = e^{ik\cdot x}[D_\mu(\mathscr{A})+ik_\mu]$$

and this leads to:

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{-\epsilon\{[D(\mathscr{A})+ik]^2+\frac{ie}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

$$k_\mu \to \frac{1}{\sqrt{\epsilon}}k_\mu$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{-\{[\sqrt{\epsilon}D(\mathscr{A})+ik]^2+\frac{ie}{2}\epsilon\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{-\{[\sqrt{\epsilon}D(\mathscr{A})+ik]^2+\frac{ie}{2}\epsilon\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

Now we take the limit $\epsilon \longrightarrow 0$ remembering that

$$\mathrm{Tr}\,\gamma_5 = 0, \qquad \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\right) = 0, \qquad \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = -4\epsilon^{\mu\nu\alpha\beta}.$$

Thus, the first term contributing in the ϵ-expansion is the one with **four** γ^μ's

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\left[\frac{1}{2}\left(\frac{ie\epsilon}{2}\right)^2\int d^4x\beta(x)\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right)\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}e^{k^2}\right.$$
$$\left. + \mathcal{O}(\epsilon^4)\right]$$

$\epsilon \longrightarrow 0$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = \frac{ie^2}{16\pi^2}\int d^4x\,\beta(x)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{-\{[\sqrt{\epsilon}D(\mathscr{A})+ik]^2+\frac{ie}{2}\epsilon\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

Now we take the limit $\epsilon \longrightarrow 0$ remembering that

$$\mathrm{Tr}\,\gamma_5 = 0, \qquad \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\right) = 0, \qquad \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = -4\epsilon^{\mu\nu\alpha\beta}.$$

Thus, the first term contributing in the ϵ-expansion is the one with **four** γ^μ's

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\cancel{\epsilon^2}}\left[\frac{1}{2}\left(\frac{ie\cancel{\epsilon}}{2}\right)^2\int d^4x\beta(x)\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right)\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}e^{k^2} + \mathcal{O}(\epsilon^4)\right]$$

$$k^2 = -\delta_{\mu\nu}k^\mu k^\nu$$

$$\Big\Downarrow \quad \epsilon \longrightarrow 0$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = \frac{ie^2}{16\pi^2}\int d^4x\,\beta(x)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = \frac{ie^2}{16\pi^2}\int d^4x\,\beta(x)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

Since the anomaly is given by

$$\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = -2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}$$

we arrive at the known Adler-Bell-Jackiw anomaly in **Euclidean space**

$$\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = \frac{ie^2}{16\pi^2}\int d^4x\,\beta(x)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

$$\boxed{\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = \frac{ie^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)}$$

mind the i !

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{ie^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

With just a few changes, this calculation is easily generalized to the case of the **singlet anomaly**

- Take the vector gauge field $\mathcal{V}_\mu(x)$ to be non-Abelian, while keeping $\mathcal{A}_\mu(x)$Abelian

- Add group theory traces in all expressions

- Set at the end $\mathcal{V}_\mu(x) = g\mathscr{A}_\mu(x)$ and $\mathcal{A}_\mu(x) = 0$

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{ig^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)\right]$$

mind the i !
(again)

The connection with topology

From our previous discussion, we know that the axial anomaly is given by (taking $\beta(x)$ = constant)

$$\int d^4x\, \partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i \lim_{\epsilon\to 0} {\rm Tr} \left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\}$$

Instead of a basis of plane waves, to compute the right-hand side we can use a basis of **eigenfunctions of the Dirac operator**

$$i\not{D}(\mathscr{A})\psi_n(x) = \lambda_n \psi_n(x)$$

But the Dirac operator anticommutes with the chirality matrix γ_5

$$i\not{D}(\mathscr{A})\gamma_5 = -\gamma_5 i\not{D}(\mathscr{A})$$

$$\Downarrow$$

$$i\not{D}(\mathscr{A})\gamma_5\psi_n(x) = -\gamma_5 i\not{D}(\mathscr{A})\psi_n(x) = -\lambda_n\gamma_5\psi_n(x)$$

$$i\not{D}(\mathscr{A})\psi_n(x) = \lambda_n\psi_n(x) \qquad\qquad i\not{D}(\mathscr{A})\gamma_5\psi_n(x) = -\lambda_n\gamma_5\psi_n(x)$$

For each eigenstate with $\lambda_n > 0$ there is another eigenstate of **opposite eigenvalue** $-\lambda_n < 0$

All **nonzero** eigenvectors of the Dirac operators are **paired**!

Moreover, since they have different (nonzero) eigenvalues, $\psi_n(x)$ and $\gamma_5\psi_n(x)$ are **orthogonal** (the Dirac operator is self-adjoint)

$$\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x) = 0 \qquad\qquad (\lambda_n \neq 0)$$

$$\int d^4x\, \overline{\psi}_n(x)\gamma_5\psi_n(x) = 0 \qquad\qquad (\lambda_n \neq 0)$$

$$\Downarrow$$

$$-2i\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[i\not{D}(\mathscr{A})]^2}\right\} = -2i\sum_{\lambda_n\neq 0}\int d^4x\, e^{-\epsilon\lambda_n^2}\overline{\psi}_n(x)\gamma_5\psi_n(x)$$

$$-2i\sum_{\lambda_n=0}\int d^4x\, \overline{\psi}_n(x)\gamma_5\psi_n(x)$$

In the limit $\epsilon \longrightarrow 0$ the sum over nonzero eigenvalues tends to zero (due to orthogonality)

Thus, the anomaly is only determined by the **zero modes** of the Dirac operator

$$\int d^4x\, \partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = -2i\sum_{\lambda_n=0}\int d^4x\, \overline{\psi}_n(x)\gamma_5\psi_n(x)$$

$$\int d^4x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\sum_{\lambda_n=0}\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x)$$

The zero modes of the Dirac operator can be classified into **positive** and **negative chirality**:

$$\gamma_5\psi_n^{(\pm)}(x) = \pm\psi_n^{(\pm)}(x) \qquad\qquad (\lambda_n = 0)$$

Then, the sum over zero modes can be written as

$$\sum_{\lambda_n=0}\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x) = \sum_{\lambda_n=0,+}\int d^4x\,\overline{\psi}{}^{(+)}_n(x)\psi^{(+)}_n(x) - \sum_{\lambda_n=0,-}\int d^4x\,\overline{\psi}{}^{(-)}_n(x)\psi^{(-)}_n(x)$$

and since the states are normalized

$$\sum_{\lambda_n=0}\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x)$$
$$= (\#\text{ of }+\text{'ve chirality zero modes}) - (\#\text{ of }-\text{'ve chirality zero modes})$$

$$\int d^4x\, \partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i \sum_{\lambda_n=0} \int d^4x\, \overline{\psi}_n(x)\gamma_5\psi_n(x)$$

The zero modes of the Dirac operator can be classified into **positive** and **negative chirality**:

$$\gamma_5\psi_n^{(\pm)}(x) = \pm\psi_n^{(\pm)}(x) \qquad\qquad (\lambda_n = 0)$$

Then, the sum over zero modes can be written as

$$\sum_{\lambda_n=0}\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x) = \sum_{\lambda_n=0,+}\overbrace{\int d^4x\,\overline{\psi}{}_n^{(+)}(x)\psi_n^{(+)}(x)}^{=1} - \sum_{\lambda_n=0,-}\overbrace{\int d^4x\,\overline{\psi}{}_n^{(-)}(x)\psi_n^{(-)}(x)}^{=1}$$

and since the states are normalized

$$\sum_{\lambda_n=0}\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x)$$
$$= (\#\text{ of }+\text{'ve chirality zero modes}) - (\#\text{ of }-\text{'ve chirality zero modes})$$

Thus, the integrated axial anomaly is given by the difference between the number of **zero modes** of the Dirac operator with positive and negative chirality:

$$\int d^4x\, \partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\Big(n_+ - n_-\Big)$$

In fact, we can define the operators

$$D_\pm \equiv i\not{D}(\mathscr{A})P_\pm \qquad \text{where} \qquad P_\pm = \frac{1}{2}\Big(\mathbb{I} \pm \gamma_5\Big)$$

we can write

$$n_+ = \dim\ker D_+ \qquad\qquad n_- = \dim\ker D_-$$

Using $i\not{D}(\mathscr{A})P_\pm = P_\mp i\not{D}(\mathscr{A})$ and **self-adjointness** of the Dirac operator,

$$D_- = D_+^\dagger \quad \longrightarrow \quad n_- = \dim\ker D_+^\dagger$$

$$D_\pm \equiv i\not{D}(\mathscr{A})P_\pm$$

With all this we have arrived at the result

$$\boxed{\int d^4x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\Big(\dim\ker D_+ - \dim\ker D_+^\dagger\,\Big)}$$

The term inside the bracket on the right-hand side is known in Mathematics as the **index of the operator** D_+

$$\boxed{\int d^4x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\,(\mathrm{ind}\,D_+)}$$

In fact, the analysis is valid not on in $D = 4$ but for **any dimension** $D = 2n$

$$\boxed{\int d^{2n}x\, \partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = -2i\,(\mathrm{ind}\, D_+)}$$

This index only depends on **topological properties** of the manifold and the external gauge field $\mathscr{A}_\mu(x)$

A short (and non-sophisticated) excursion into Mathematics

Let us consider a nonabelian gauge theory defined on a **Euclidean closed even-dimensional manifold** M.

The gauge **connection** defines a **one-form** field taking values in the Lie algebra

$$\mathscr{A} = \mathscr{A}_\mu dx^\mu \qquad \text{where} \qquad \mathscr{A}_\mu = \mathscr{A}_\mu^a T^a$$

we should keep in mind that

$$\boxed{[T^a, T^b] = i f^{abc} T^c}$$

$$\mathscr{A} \wedge \mathscr{A} = T^a T^b \mathscr{A}^a \wedge \mathscr{A}^b = \frac{1}{2}[T^a, T^b]\mathscr{A}^a \wedge \mathscr{A}^b = \frac{i}{2} f^{abc} \mathscr{A}^a \wedge \mathscr{A}^b T^c$$

The **field strength** is a **two-form** given by

$$\mathscr{F} = d\mathscr{A} + \mathscr{A} \wedge \mathscr{A} \qquad \text{with} \qquad \mathscr{F} = \frac{1}{2}\mathscr{F}_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$\boxed{\mathscr{F}_{\mu\nu}^a = \partial_\mu \mathscr{A}_\nu^a - \partial_\nu \mathscr{A}_\mu^a + i f^{abc} \mathscr{A}^b \mathscr{A}^c}$$

Under gauge transformations, the connection transforms as

$$\mathscr{A} \longrightarrow g^{-1}dg + g^{-1}\mathscr{A}\,g$$

while the field strength two-form transforms in the adjoint representation of the gauge group

$$\mathscr{F} \longrightarrow g^{-1}\mathscr{F}\,g$$

Finally, from $\mathscr{F} = d\mathscr{A} + \mathscr{A}\wedge\mathscr{A}$

$$d\mathscr{F} = d\mathscr{A}\wedge\mathscr{A} - \mathscr{A}\wedge d\mathscr{A} \xrightarrow{\;d\mathscr{A} = \mathscr{F} - \mathscr{A}\wedge\mathscr{A}\;} \begin{aligned} d\mathscr{F} &= \mathscr{F}\wedge\mathscr{A} - \mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A} \\ &\quad -\mathscr{A}\wedge\mathscr{F} + \mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A} \end{aligned}$$

we get the **Bianchi identity**

$$d\mathscr{F} - \mathscr{F}\wedge\mathscr{A} + \mathscr{A}\wedge\mathscr{F} = 0$$

We want to investigate the properties of **invariant polynomials** of the form ($\dim M = 2m$)

$$P(\mathscr{F}) = \sum_{n+j\leq m} c_{n,j}\Big(\mathrm{Tr}\,\mathscr{F}^n\Big)^j \qquad \begin{array}{l} c_{n,j}\in\mathbb{C} \\ \mathscr{F}^n \equiv \mathscr{F}\wedge \overset{n}{\ldots} \wedge\mathscr{F}\end{array}$$

- The polynomial is **gauge invariant**: $P(g\mathscr{F}g^{-1}) = P(\mathscr{F})$

$$\mathrm{Tr}\,\mathscr{F}^n \longrightarrow \mathrm{Tr}\left(g\mathscr{F}^n g^{-1}\right) = \mathrm{Tr}\,\mathscr{F}^n$$

- It is **closed**: $dP(\mathscr{F}) = 0$

$$d\mathrm{Tr}\,\mathscr{F}^n = \mathrm{Tr}\left(d\mathscr{F}\wedge\ldots\wedge\mathscr{F}\right) + \ldots + \mathrm{Tr}\left(\mathscr{F}\wedge\ldots\wedge d\mathscr{F}\right) = n\mathrm{Tr}\left(d\mathscr{F}\,\mathscr{F}^{n-1}\right)$$

using the Bianchi identity $d\mathscr{F} - \mathscr{F}\wedge\mathscr{A} + \mathscr{A}\wedge\mathscr{F} = 0$

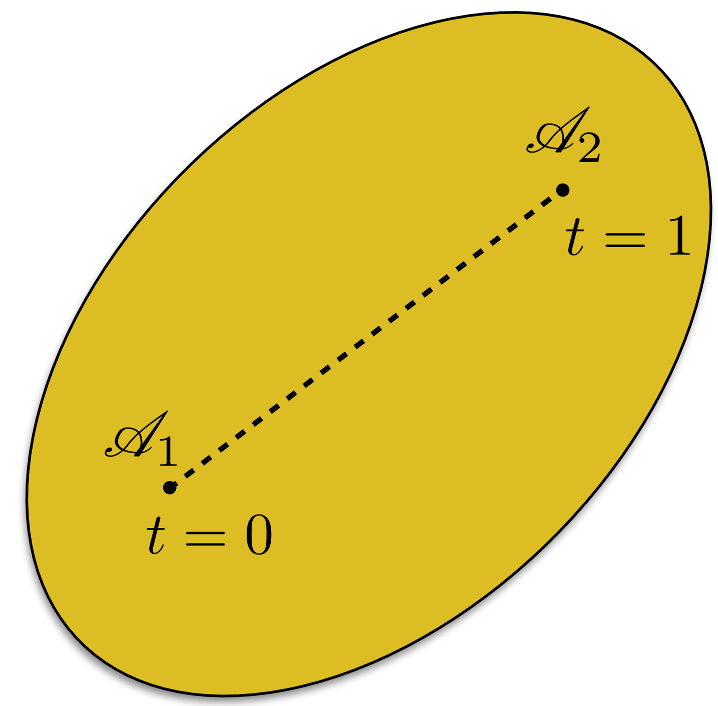
$$d\mathrm{Tr}\,\mathscr{F}^n = n\mathrm{Tr}\left(\mathscr{F}\mathscr{A}\,\mathscr{F}^{n-1}\right) - n\mathrm{Tr}\left(\mathscr{A}\,\mathscr{F}^n\right) = 0$$

- $\int_{M_{2n}} \mathrm{Tr}\, \mathscr{F}^n$ is invariant under **deformations** of the connection

Let us consider a continuous family of connections joining $\mathscr{A}_1$ and $\mathscr{A}_2$

$$\mathscr{A}_t = (1-t)\mathscr{A}_1 + t\mathscr{A}_2 \qquad\qquad (0 \le t \le 1)$$

$$\mathscr{A}_t \longrightarrow g^{-1}dg + g^{-1}\mathscr{A}_t g$$



Defining $\mathscr{F}_t = d\mathscr{A}_t + \mathscr{A}_t \wedge \mathscr{A}_t$ we compute

$$\frac{\partial}{\partial t}\mathrm{Tr}\, \mathscr{F}_t^n = n\mathrm{Tr}\left(\dot{\mathscr{F}}_t \mathscr{F}_t^{n-1}\right)$$

Now we can use $\dot{\mathscr{F}}_t = d\dot{\mathscr{A}}_t + \dot{\mathscr{A}}_t \wedge \mathscr{A}_t + \mathscr{A}_t \wedge \dot{\mathscr{A}}_t$ to write

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Tr}\, \mathscr{F}_t^n &= n\mathrm{Tr}\left(d\dot{\mathscr{A}}_t \mathscr{F}_t^{n-1}\right) + n\mathrm{Tr}\left(\dot{\mathscr{A}}_t \mathscr{A}_t \mathscr{F}_t^{n-1}\right) + n\mathrm{Tr}\left(\mathscr{A}_t \dot{\mathscr{A}}_t \mathscr{F}_t^{n-1}\right) \\ &= n\mathrm{Tr}\left(d\dot{\mathscr{A}}_t \mathscr{F}_t^{n-1}\right) + n\mathrm{Tr}\left(\dot{\mathscr{A}}_t \mathscr{A}_t \mathscr{F}_t^{n-1}\right) - n\mathrm{Tr}\left(\dot{\mathscr{A}}_t \mathscr{F}_t^{n-1} \mathscr{A}_t\right)\end{aligned}$$

$$\frac{\partial}{\partial t}\mathrm{Tr}\,\mathscr{F}_t^n = n\mathrm{Tr}\Big(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\Big) + n\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right) - n\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\mathscr{A}_t\right)$$

Applying the Bianchi identity recursively, one can easily prove

$$\frac{\partial}{\partial t}\mathrm{Tr}\,\mathscr{F}_t^n = nd\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)$$

$$\boxed{d\mathscr{F} - \mathscr{F}\wedge\mathscr{A} + \mathscr{A}\wedge\mathscr{F} = 0}$$

Integrating over the parameter t shows that

$$\mathrm{Tr}\,\mathscr{F}_2^n - \mathrm{Tr}\,\mathscr{F}_1^n = nd\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right) \equiv d\int_0^1 dt\,Q_{2n-1}^0(\mathscr{A}_t,\mathscr{F}_t)$$

Thus, given any closed $2n$-dimensional surface submanifold $M_{2n}\subset M$

$$\int_{M_{2n}}\mathrm{Tr}\,\mathscr{F}_1^n = \int_{M_{2n}}\mathrm{Tr}\,\mathscr{F}_2^n$$

and the result of the integral is **independent of the connection**.

$$d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t d\mathscr{F}_t\mathscr{F}_t^{n-2}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t d\mathscr{F}_t\mathscr{F}_t^{n-3}\right)\ldots-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}d\mathscr{F}_t\right)$$

$$d\mathscr{F}_t=\mathscr{F}_t\wedge\mathscr{A}_t+\mathscr{A}_t\wedge\mathscr{F}_t$$

$$\begin{aligned}d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=&\ \mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t\mathscr{A}_t\mathscr{F}_t^{n-2}\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right)\\&-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{2}\mathscr{A}_t\mathscr{F}_t^{n-3}\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t\mathscr{A}_t\mathscr{F}_t^{n-2}\right)\\&-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{3}\mathscr{A}_t\mathscr{F}_t^{n-4}\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{2}\mathscr{A}_t\mathscr{F}_t^{n-3}\right)+\ldots\\&\ldots-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\mathscr{A}_t\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}\mathscr{A}_t\mathscr{F}_t\right)\end{aligned}$$

$$d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}\mathscr{A}_t\mathscr{F}_t\right)$$

$$d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t d\mathscr{F}_t\mathscr{F}_t^{n-2}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t d\mathscr{F}_t\mathscr{F}_t^{n-3}\right)\ldots-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}d\mathscr{F}_t\right)$$

$$d\mathscr{F}_t=\mathscr{F}_t\wedge\mathscr{A}_t+\mathscr{A}_t\wedge\mathscr{F}_t$$

$$\begin{aligned}d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=&\ \mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)\cancel{-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t\mathscr{A}_t\mathscr{F}_t^{n-2}\right)}+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right)\\&-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{2}\mathscr{A}_t\mathscr{F}_t^{n-3}\right)+\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t\mathscr{A}_t\mathscr{F}_t^{n-2}\right)}\\&-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{3}\mathscr{A}_t\mathscr{F}_t^{n-4}\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{2}\mathscr{A}_t\mathscr{F}_t^{n-3}\right)+\ldots\\&\ldots-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\mathscr{A}_t\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}\mathscr{A}_t\mathscr{F}_t\right)\end{aligned}$$

$$d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}\mathscr{A}_t\mathscr{F}_t\right)$$

$$d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t d\mathscr{F}_t\mathscr{F}_t^{n-2}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t d\mathscr{F}_t\mathscr{F}_t^{n-3}\right)\ldots-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}d\mathscr{F}_t\right)$$

$$d\mathscr{F}_t=\mathscr{F}_t\wedge\mathscr{A}_t+\mathscr{A}_t\wedge\mathscr{F}_t$$

$$\begin{aligned}d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)&=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)-\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t\mathscr{A}_t\mathscr{F}_t^{n-2}\right)}+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right)\\&-\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{2}\mathscr{A}_t\mathscr{F}_t^{n-3}\right)}+\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t\mathscr{A}_t\mathscr{F}_t^{n-2}\right)}\\&-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{3}\mathscr{A}_t\mathscr{F}_t^{n-4}\right)+\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{2}\mathscr{A}_t\mathscr{F}_t^{n-3}\right)}+\ldots\\&\ldots-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\mathscr{A}_t\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}\mathscr{A}_t\mathscr{F}_t\right)\end{aligned}$$

$$d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}\mathscr{A}_t\mathscr{F}_t\right)$$

$$d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t d\mathscr{F}_t\mathscr{F}_t^{n-2}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t d\mathscr{F}_t\mathscr{F}_t^{n-3}\right)\ldots-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}d\mathscr{F}_t\right)$$

$$\Big\Downarrow \quad d\mathscr{F}_t=\mathscr{F}_t\wedge\mathscr{A}_t+\mathscr{A}_t\wedge\mathscr{F}_t$$

$$\begin{aligned}
d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)&-\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t\mathscr{A}_t\mathscr{F}_t^{n-2}\right)}+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right)\\
&-\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^2\mathscr{A}_t\mathscr{F}_t^{n-3}\right)}+\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t\mathscr{A}_t\mathscr{F}_t^{n-2}\right)}\\
&-\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^3\mathscr{A}_t\mathscr{F}_t^{n-4}\right)}+\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^2\mathscr{A}_t\mathscr{F}_t^{n-3}\right)}+\ldots\\
&\ldots-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\mathscr{A}_t\right)+\cancel{\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}\mathscr{A}_t\mathscr{F}_t\right)}
\end{aligned}$$

$$\Big\Downarrow$$

$$d\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)=\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)+\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right)-\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-2}\mathscr{A}_t\mathscr{F}_t\right)$$

$$\frac{\partial}{\partial t}\mathrm{Tr}\,\mathscr{F}_t^n = n\mathrm{Tr}\Big(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\Big) + n\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right) - n\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\mathscr{A}_t\right)$$

Applying the Bianchi identity recursively, one can easily prove

$$\frac{\partial}{\partial t}\mathrm{Tr}\,\mathscr{F}_t^n = nd\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)$$

$$\boxed{d\mathscr{F} - \mathscr{F}\wedge\mathscr{A} + \mathscr{A}\wedge\mathscr{F} = 0}$$

Integrating over the parameter t shows that

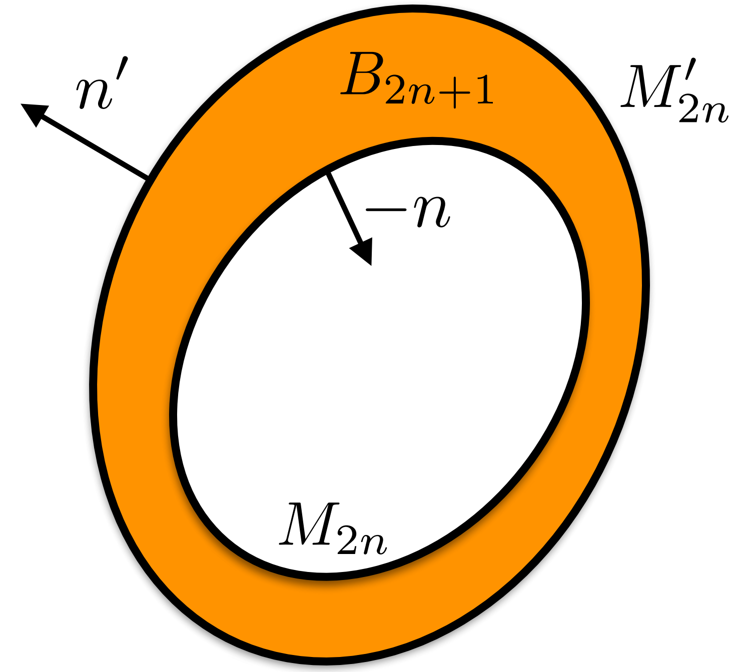
$$\mathrm{Tr}\,\mathscr{F}_2^n - \mathrm{Tr}\,\mathscr{F}_1^n = nd\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right) \equiv d\int_0^1 dt\,Q_{2n-1}^0(\mathscr{A}_t,\mathscr{F}_t)$$

Thus, given any closed $2n$-dimensional surface submanifold $M_{2n}\subset M$

$$\int_{M_{2n}}\mathrm{Tr}\,\mathscr{F}_1^n = \int_{M_{2n}}\mathrm{Tr}\,\mathscr{F}_2^n$$

and the result of the integral is **independent of the connection**.

- $\int_{M_{2n}} \mathrm{Tr}\,\mathscr{F}^n$ is invariant under deformations of the submanifold M_{2n}



Let M'_{2n} be a deformation of M_{2n} and let

$$\partial B_{2n+1} = M'_{2n} - M_{2n}$$

Then, using $d\mathrm{Tr}\,\mathscr{F}^n = 0$

$$\int_{M'_{2n}} \mathrm{Tr}\,\mathscr{F}^n - \int_{M_{2n}} \mathrm{Tr}\,\mathscr{F}^n = \int_{\partial B_{2n+1}} \mathrm{Tr}\,\mathscr{F}^n = \int_{B_{2n+1}} d\mathrm{Tr}\,\mathscr{F}^n = 0$$

and we conclude that the integral does not change under deformations of the submanifold

$$\int_{M'_{2n}} \mathrm{Tr}\,\mathscr{F}^n = \int_{M_{2n}} \mathrm{Tr}\,\mathscr{F}^n$$

We have seen how using invariant polynomials we can construct **topological invariants** both with respect to deformation of the **gauge field** and of the **manifold**.

We introduce two examples:

- **Chern classes**: given a $\mathrm{U}(n)$ gauge bundle, the **total Chern class** is defined as

$$c(\mathscr{F}) = \det\left(1 + \frac{i}{2\pi}\mathscr{F}\right)$$

to write it in terms of invariant polynomials, we notice that since $\mathscr{F}$ is Hermitian [it lives in the algebra of $\mathrm{U}(n)$], we can diagonalize it

$$\frac{i}{2\pi}\mathscr{F} = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$$

$$c(\mathscr{F}) = \det\left(1 + \frac{i}{2\pi}\mathscr{F}\right) \qquad\qquad \frac{i}{2\pi}\mathscr{F} = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$$

The total Chern class is written then as

$$c(\mathscr{F}) = \prod_{i=1}^{n}(1 + x_i) = 1 + \sum_{i=1}^{n} x_i + \sum_{i<j}^{n} x_i x_j + \ldots + \prod_{i=1}^{n} x_i$$

so we can identify the ***i*-th Chern class**

$$c_1(\mathscr{F}) = \sum_{i=1}^{n} x_i = \frac{i}{2\pi}\operatorname{Tr}\mathscr{F}$$

$$c_2(\mathscr{F}) = \sum_{i<j}^{n} x_i x_j = \frac{1}{2}\left[\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2\right] = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2\left[(\operatorname{Tr}\mathscr{F})^2 - \operatorname{Tr}\mathscr{F}^2\right]$$

$$\vdots$$

$$c_n(\mathscr{F}) = \det\left(\frac{i}{2\pi}\mathscr{F}\right)$$

- **Chern character**: given again a $\mathrm{U}(n)$ gauge bundle, we define

$$\mathrm{ch}(\mathscr{F}) = \mathrm{Tr}\,\exp\left(\frac{i}{2\pi}\mathscr{F}\right)$$

Formally expanding the exponential, we find the ***i*-th Chern character**

$$\mathrm{Tr}\,\exp\left(\frac{i}{2\pi}\mathscr{F}\right) = \sum_{k=0}^{m}\frac{1}{k!}\left(\frac{i}{2\pi}\mathscr{F}\right)^{k}$$

$$\Downarrow$$

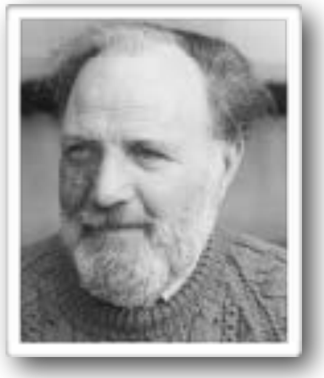
$$\mathrm{ch}_0(\mathscr{F}) = r$$

$$\mathrm{ch}_j(\mathscr{F}) = \frac{1}{j!}\left(\frac{i}{2\pi}\right)^{j}\mathrm{Tr}\,\mathscr{F}^{j} \qquad 2 \leq 2j \leq \dim M$$

Atiyah-Singer index theorem

(first version)

Sir Michael Atiyah
(b. 1929)



Isadore Singer
(b. 1924)

Let be a vector bundle defined on an ***even-dimensional* flat manifold without boundary** M.

The index of the Weyl operator $D_\pm \equiv i\not{D}(\mathscr{A})P_\pm$ is given by

$$\boxed{\mathrm{ind}\, D_+ = \int_M [\mathrm{ch}(\mathscr{F})]_{\mathrm{vol}}}$$

where $\mathrm{ch}(\mathscr{F}) = \mathrm{Tr}\, \exp\left(\frac{i}{2\pi}\mathscr{F}\right)$

In particular, if $\dim M = 2m$

$$\mathrm{ind}\, D_+ = \int_M \mathrm{ch}_m(\mathscr{F}) = \frac{1}{m!}\left(\frac{i}{2\pi}\right)^m \int_M \mathrm{Tr}\, \mathscr{F}^m$$

$$\int d^{2n}x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\,(\text{ind}\,D_+) \qquad \text{ind}\,D_+ = \int_M \text{ch}_m(\mathscr{F}) = \frac{1}{m!}\left(\frac{i}{2\pi}\right)^m \int_M \text{Tr}\,\mathscr{F}^m$$

Using the Atiyah-Singer index theorem, the axial anomaly in $D = 2n$ is given by

$$\int d^{2n}x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -\frac{2i}{n!}\left(\frac{i}{2\pi}\right)^n \int_M \text{Tr}\,\mathscr{F}^n$$

To rewrite the right-hand side, we use $\mathscr{F} = \frac{1}{2}\mathscr{F}_{\mu\nu}dx^\mu \wedge dx^\nu$

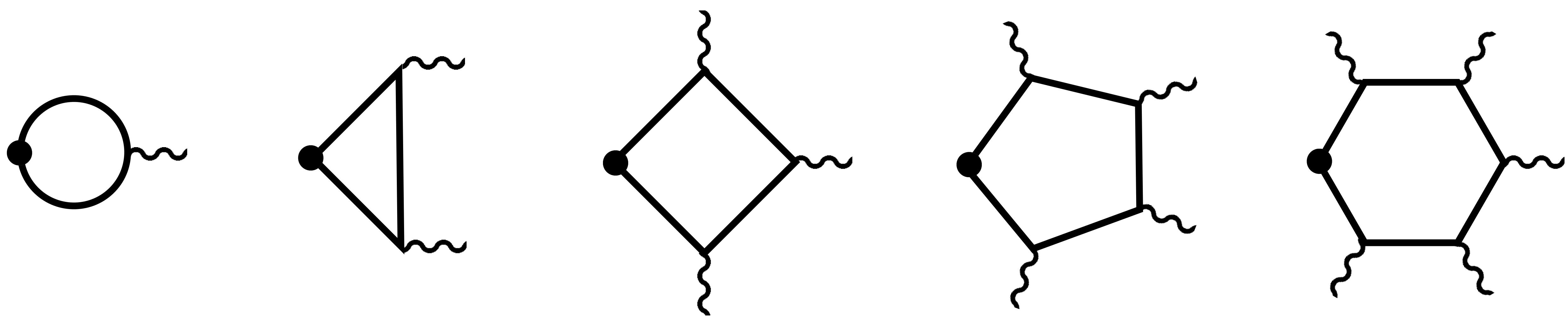
$$\begin{aligned}\mathscr{F}^n &= \frac{1}{2^n}\mathscr{F}_{\mu_1\nu_n}\ldots\mathscr{F}_{\mu_n\nu_n}dx^{\mu_1}\wedge dx^{\nu_1}\wedge\ldots\wedge dx^{\mu_n}\wedge dx^{\nu_n} \\ &= \frac{1}{2^n}\epsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\mathscr{F}_{\mu_1\nu_1}\ldots\mathscr{F}_{\mu_n\nu_n}d^{2n}x\end{aligned}$$

$$\int d^{2n}x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -\frac{2i}{n!}\left(\frac{i}{4\pi}\right)^n \int d^{2n}x\,\epsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\text{Tr}\left(\mathscr{F}_{\mu_1\nu_1}\ldots\mathscr{F}_{\mu_n\nu_n}\right)$$

$$\partial_\mu \langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = -\frac{2i}{n!}\left(\frac{i}{4\pi}\right)^n \epsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\,\mathrm{Tr}\left(\mathscr{F}_{\mu_1\nu_1}\ldots\mathscr{F}_{\mu_n\nu_n}\right)$$

The **axial anomaly** in $D = 2n$ has the following **properties**:

- It is **determined** by the one-loop, $(n+1)$-gon diagram with one **axial-vector** current and n **vector currents**

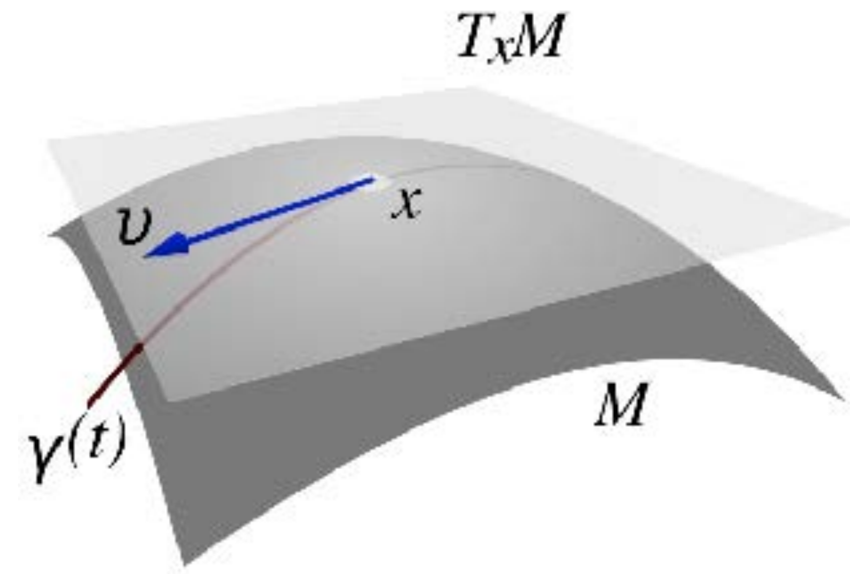


$D = 2$ $\quad$ $D = 4$ $\quad$ $D = 6$ $\quad$ $D = 8$ $\quad$ $D = 10$

- The anomaly is **exact** at one loop.

But remember that **gravity** also contributes to the axial anomaly…

On the $2n$-dimensional Euclidean manifold, we have the **freedom** to choose an **orthonormal basis** of the tangent (and cotangent) space at each point **independently**



$$TM_x: \quad \mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$$

$$TM_x^*: \quad \theta^a(\mathbf{e}_b) = \delta^a_b$$

These relations are left invariant by $\mathrm{SO}(2n)$ rotations of the frame. It is with respect to these transformations that **Dirac spinors** are defined:

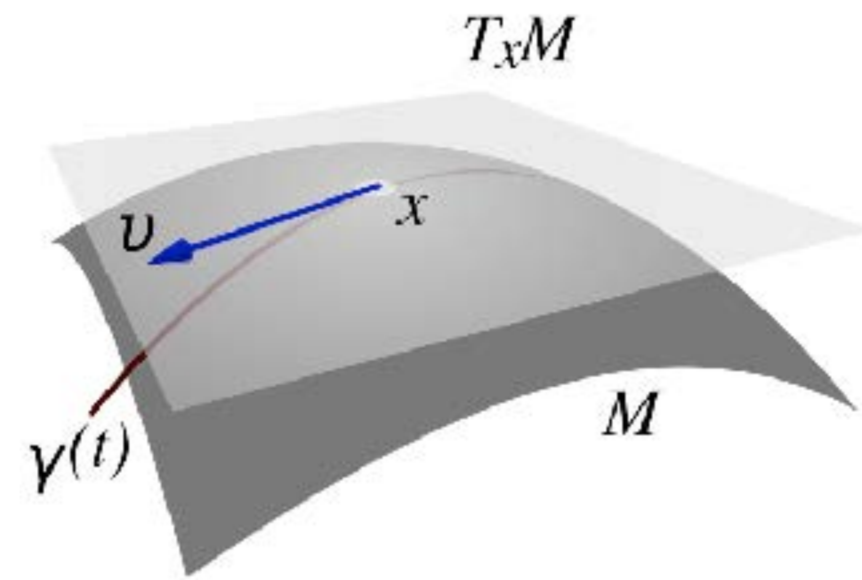
$$\{\gamma^a, \gamma^b\} = -2\delta^{ab}\mathbb{I} \qquad \gamma^{a\dagger} = -\gamma^a$$

$$\sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$$

$$\psi'(x) = e^{-\frac{i}{2}\xi_{ab}(x)\sigma^{ab}}\psi(x)$$

General relativity can be seen as a $\mathrm{SO}(2n)$ **gauge theory** for the choice of **local frames** [$\Rightarrow$ $\mathrm{SO}(1{,}2n{-}1)$ in Lorentzian signature]

To define the notion of **parallel transport** along a curve γ(t) we introduce the **one-form spin connection** ω_{ab}



For a general field $\Phi(x)$ transforming in some representation Σ^{ab} of the local $\mathrm{SO}(2n)$ group,

$$\nabla_{\mathbf{v}}\Phi = d\Phi(\mathbf{v}) + \frac{1}{2}\omega_{ab}(\mathbf{v})\Sigma^{ab}\Phi$$

$$\nabla_{\mathbf{v}}\Phi = 0$$

In the case of a **spinor**, the representation is $\Sigma^{ab} \equiv \sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$ and

$$\nabla_{\mathbf{v}}\psi = d\psi(\mathbf{v}) + \frac{1}{2}\omega_{ab}(\mathbf{v})\sigma^{ab}\psi$$

In terms of the spin connection, the **curvature two-form** is defined by

$$\mathscr{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

Taking the exterior derivative

$$d\mathscr{R}^a{}_b = d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b$$

and using

$$d\omega^a{}_b = \mathscr{R}^a{}_b - \omega^a{}_c \wedge \omega^c{}_b$$

we arrive at the **Bianchi identity**

$$d\mathscr{R}^a{}_b - \mathscr{R}^a{}_c \wedge \omega^c{}_b + \omega^a{}_c \wedge \mathscr{R}^c{}_b = 0$$

Gauge theories

$$\mathscr{F} = d\mathscr{A} + \mathscr{A} \wedge \mathscr{A}$$

$$d\mathscr{F} - \mathscr{F} \wedge \mathscr{A} + \mathscr{A} \wedge \mathscr{F} = 0$$

Gravity

$$\mathscr{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

$$d\mathscr{R}^a{}_b - \mathscr{R}^a{}_c \wedge \omega^c{}_b + \omega^a{}_c \wedge \mathscr{R}^c{}_b = 0$$

The curvature two-form can be expressed in the basis of differentials

$$\mathscr{R}^a{}_b = \frac{1}{2}\mathscr{R}^a{}_{b,\mu\nu} dx^\mu \wedge dx^\nu$$

Lorentz indices can be turned into Einstein ones by using the **vielbein**

$$\mathbf{e}_a = e_a{}^\mu(x)\partial_\mu \qquad \partial_\mu = e_\mu{}^a(x)\mathbf{e}_a$$

which satisfy

$$\delta_{ab} = e_a{}^\mu(x) e_b{}^\nu(x) g_{\mu\nu}(x) \qquad g_{\mu\nu}(x) = e_\mu{}^a(x) e_\nu{}^b(x)\eta_{ab}$$

In terms of the vielbein, the Einstein components of the curvature tensor are

$$\mathscr{R}^\mu{}_{\nu,\alpha\beta} = e_a{}^\mu e_\nu{}^b \mathscr{R}^a{}_{b,\alpha\beta}$$

Given the transformation properties of $\omega^a{}_b$ and $\mathscr{R}^a{}_b$

$$\omega \longrightarrow U^{-1}dU + U^{-1}\omega U \qquad\qquad \mathscr{R} \longrightarrow U^{-1}\mathscr{R}\, U$$

We can define invariant polynomials as we did with gauge theories

$$P(\mathscr{R}) = \sum_{n+j\leq m} a_{n,j}\Big(\mathrm{Tr}\,\mathscr{R}^n\Big)^j$$

$$a_{n,j} \in \mathbb{R}$$

$$(\mathscr{R}^n)^a{}_b = \mathscr{R}^a{}_{c_1} \wedge \mathscr{R}^{c_1}{}_{c_2} \wedge \ldots \wedge \mathscr{R}^{c_{n-1}}{}_b$$

where the trace is over $\mathrm{SO}(2n)$ indices.

- The polynomials are invariant under $\mathrm{SO}(2n)$ transformations:

$$P(\mathscr{R}) = P(U^{-1}\mathscr{R}U)$$

- They are closed:

$$dP(\mathscr{R}) = 0$$

- The integrals $\displaystyle\int_{M_{2m}} \mathrm{Tr}\,\mathscr{R}^m$ are topological invariants.

- The first invariant polynomial we define is the **Pontrjagin index**

$$p(\mathscr{R}) = \det\left(1 + \frac{1}{2\pi}\mathscr{R}\right)$$

The curvature two-form takes values in the Lie algebra of $\mathrm{SO}(2n)$, i.e. it is an **antisymmetric** matrix. To diagonalize it requires a complex transformation.

However, there exist **real** similarity transformations bringing the curvature to the form

$$\frac{1}{2\pi}\mathscr{R} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix} \qquad x_i \in \mathbb{R}$$

then

$$p(\mathscr{R}) = \prod_{i=1}^{n}(1 + x_i^2) = 1 + \sum_{i=1}^{n} x_i^2 + \sum_{i<j}^{n} x_i^2 x_j^2 + \ldots + \prod_{i=1}^{n} x_i^2$$

To write the Pontrjagin index in a more useful form, we notice

$$\frac{1}{2\pi}\mathscr{R} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$\operatorname{Tr}\left(\frac{1}{2\pi}\mathscr{R}\right)^{2k} = 2(-1)^k\sum_{i=1}^{n} x_i^{2k}$$

writing

$$p(\mathscr{R}) = 1 + p_1(\mathscr{R}) + p_2(\mathscr{R}) + \ldots + p_n(\mathscr{R})$$

with

$$p_1(\mathscr{R}) = \sum_{i=1}^{n} x_i^2 = -\frac{1}{8\pi^2}\operatorname{Tr}\mathscr{R}^2$$

$$p_2(\mathscr{R}) = \sum_{i<j}^{n} x_i^2 x_j^2 = \frac{1}{2}\left[\left(\sum_{i=1}^{n} x_i^2\right)^2 - \sum_{i=1}^{n} x_i^4\right] = \frac{1}{128\pi^4}\left[(\operatorname{Tr}\mathscr{R}^2)^2 - 2\operatorname{Tr}\mathscr{R}^4\right]$$

$$\vdots$$

$$p_n(\mathscr{R}) = \prod_{i=1}^{n} x_i^2 = \left(\frac{1}{2\pi}\right)^n \det\mathscr{R}$$

- The **Â-genus (A-roof)** is defined as

$$\widehat{A}(M)=\prod_{i=1}^{n}\frac{x_i/2}{\sinh(x_i/2)}=1-\frac{1}{24}\prod_{i=1}^{n}x_i^2+\frac{7}{5760}\prod_{i=1}^{n}x_i^4+\ldots$$

or using again $\mathrm{Tr}\left(\frac{1}{2\pi}\mathscr{R}\right)^{2k}=2(-1)^k\sum_{i=1}^{n}x_i^{2k}$

$$\widehat{A}(M)=1+\frac{1}{(4\pi)^2}\frac{1}{12}\mathrm{Tr}\,\mathscr{R}^2+\frac{1}{(4\pi)^4}\left[\frac{1}{288}(\mathrm{Tr}\mathscr{R}^2)^2+\frac{1}{360}\mathrm{Tr}\,\mathscr{R}^4\right]+\ldots$$

- **Euler class**

$$e(M)=\prod_{i=1}^{n}x_i$$

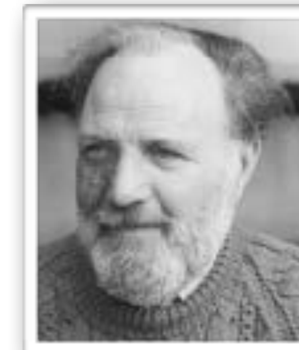
E.g., in a four-dimensional manifold, this is the "square root" of $p_2(\mathscr{R})$

$$p_2(\mathscr{R})=e(M)\wedge e(M)$$

Atiyah-Singer index theorem

(second version)

Sir Michael Atiyah
(b. 1929)



Isadore Singer
(b. 1924)

Let be a vector bundle defined on an **even-dimensional *curved manifold* without boundary** M.

The index of the Weyl operator $D_\pm \equiv i\not{D}(\mathscr{A})P_\pm$ is given now in terms of the Chern class and the Â-genus as

$$\boxed{\operatorname{ind} D_+ = \int_M [\widehat{A}(M)\operatorname{ch}(\mathscr{F})]_{\text{vol}}}$$

In four dimensions, the index has two contributions

$$\operatorname{ind} D_+ = -\frac{1}{8\pi^2}\int_M \left(\operatorname{Tr}\mathscr{F}^2 + \frac{r}{12}\operatorname{Tr}\mathscr{R}^2\right)$$

Global anomalies

So far, we have considered **anomalies** with respect to **infinitesimal** gauge transformations…

In **compactified four-dimensional Euclidean space**, gauge transformations are maps

$$g(x) : S^4 \longrightarrow \mathscr{G}$$

Then, the topology of gauge transformations is classified by the **fourth homotopy group** of the gauge group, $\pi_4(\mathscr{G})$

For some "popular groups", we have

$$\pi_4[\mathrm{SU}(3)] = 0$$

$$\pi_4[\mathrm{SU}(2)] = \mathbb{Z}_2$$

$$\pi_4[\mathrm{U}(1)] = 0$$

Thus, in the **standard model**, we can have transformations of SU(2) which are not contractible to the identity (they **wrap once** around the gauge group).

"Large" gauge transformations are important. They are **not taken care** of by the **Fadeev-Popov** trick in the functional integral. E.g., for SU(2)

Since the space of connections is contractible:

$$\int \mathscr{D}\mathscr{A}_\mu e^{-\frac{1}{4}\int d^4x \mathrm{Tr}\, \mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}} \longrightarrow$$ overcount by a factor of 2

In the **absence** of chiral fermions this is **harmless**, since the factor cancel out in expectation values.

In the case of a **Dirac fermion**

$$Z = \int \mathscr{D}\mathscr{A}_\mu \int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, e^{-\int d^4x\left(\frac{1}{4}\mathrm{Tr}\,\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}+\overline{\psi}i\not{D}\psi\right)}$$

$$= \int \mathscr{D}\mathscr{A}_\mu \det[i\not{D}(\mathscr{A})]e^{-\frac{1}{4}\int d^4x\, \mathrm{Tr}\,\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}}$$

No problem: the determinant of the Dirac operator can be defined unambiguously and the result is gauge invariant.

What about a $\mathrm{SU}(2)$ gauge theory with fundamental **chiral** fermions?

(Witten, 1982)

Let us decompose the Dirac fermion into two Weyl spinors

Edward Witten, (b. 1951)

$$\psi = \psi_+ + \psi_-$$

and write ψ_- in terms of a charge-conjugated spinor

$$\psi = \psi_+ + (\chi_+)^c$$

The Dirac action is now

$$\int d^4x\,\overline{\psi} i\not{D}\psi = \int d^4x \left[\overline{\psi}_+ i\not{D}\psi_+ + \overline{(\chi_+)^c} i\not{D}(\chi_+)^c\right]$$

But since the **fundamental** representation of $\mathrm{SU}(2)$ is **real** we can drop the charge conjugation symbol

$$\int d^4x\,\overline{\psi} i\not{D}\psi = \int d^4x \left(\overline{\psi}_+ i\not{D}\psi_+ + \overline{\chi}_+ i\not{D}\chi_+\right)$$

$$\int d^4x\,\overline{\psi} i\not{D}\psi = \int d^4x\left(\overline{\psi}_+ i\not{D}\psi_+ + \overline{\chi}_+ i\not{D}\chi_+\right)$$

Then, a Dirac fermion in the fundamental of $\mathrm{SU}(2)$ is equivalent to two positive chirality Weyl fermions.

As a consequence,

$$\det(i\not{D}) = \int \mathscr{D}\overline{\psi}_+\mathscr{D}\psi_+ \int \mathscr{D}\overline{\chi}_+\mathscr{D}\chi_+\, e^{-\int d^4x\left(\overline{\psi}_+ i\not{D}\psi_+ + \overline{\chi}_+ i\not{D}\chi_+\right)}$$

$$= \int \mathscr{D}\overline{\psi}_+\mathscr{D}\psi_+ e^{-\int d^4x\,\overline{\psi}_+ i\not{D}\psi_+} \int \mathscr{D}\overline{\chi}_+\mathscr{D}\chi_+\, e^{-\int d^4x\,\overline{\chi}_+ i\not{D}\chi_+}$$

and we arrive at

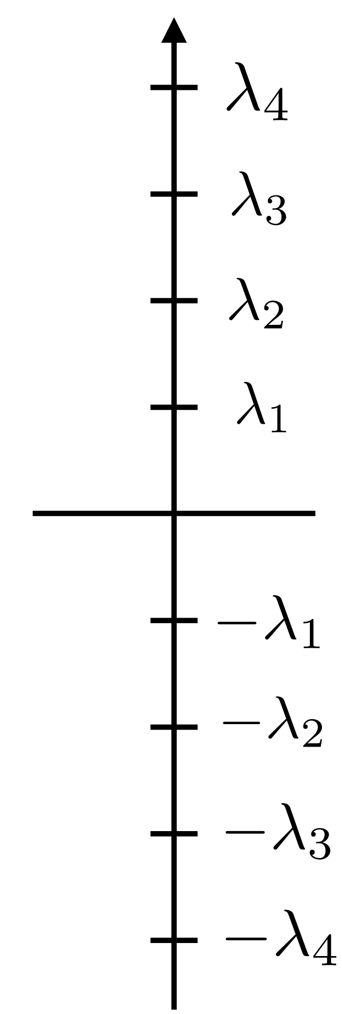
$$\int \mathscr{D}\overline{\psi}_+\mathscr{D}\psi_+ e^{-\int d^4x\,\overline{\psi}_+ i\not{D}\psi_+} = \pm[\det i\not{D}]^{\frac{1}{2}}$$

$$\int \mathscr{D}\overline{\psi}_+ \mathscr{D}\psi_+ e^{-\int d^4x\,\overline{\psi}_+ i\not{D}\psi_+} = \pm[\det(i\not{D})]^{\frac{1}{2}}$$

There is an **ambiguity** in the sign of the square root but, can we **fix** it once and for all?

Let us take a gauge field for which the Dirac operator has **no zero modes** [in other words, $\det(i\not{D}) \neq 0$]. Then, the **square root** can be defined as

$$[\det(i\not{D})]^{\frac{1}{2}} = \prod_{\lambda_n>0} \lambda_n$$



Remember: the eigenvalues of the Dirac operator are **paired** ($\lambda_n, -\lambda_n$)

Now we consider a family of connections

$$\mathscr{A}^t_\mu = (1-t)\mathscr{A}_\mu + t\mathscr{A}^U_\mu \qquad\qquad 0 \leq t \leq 1$$

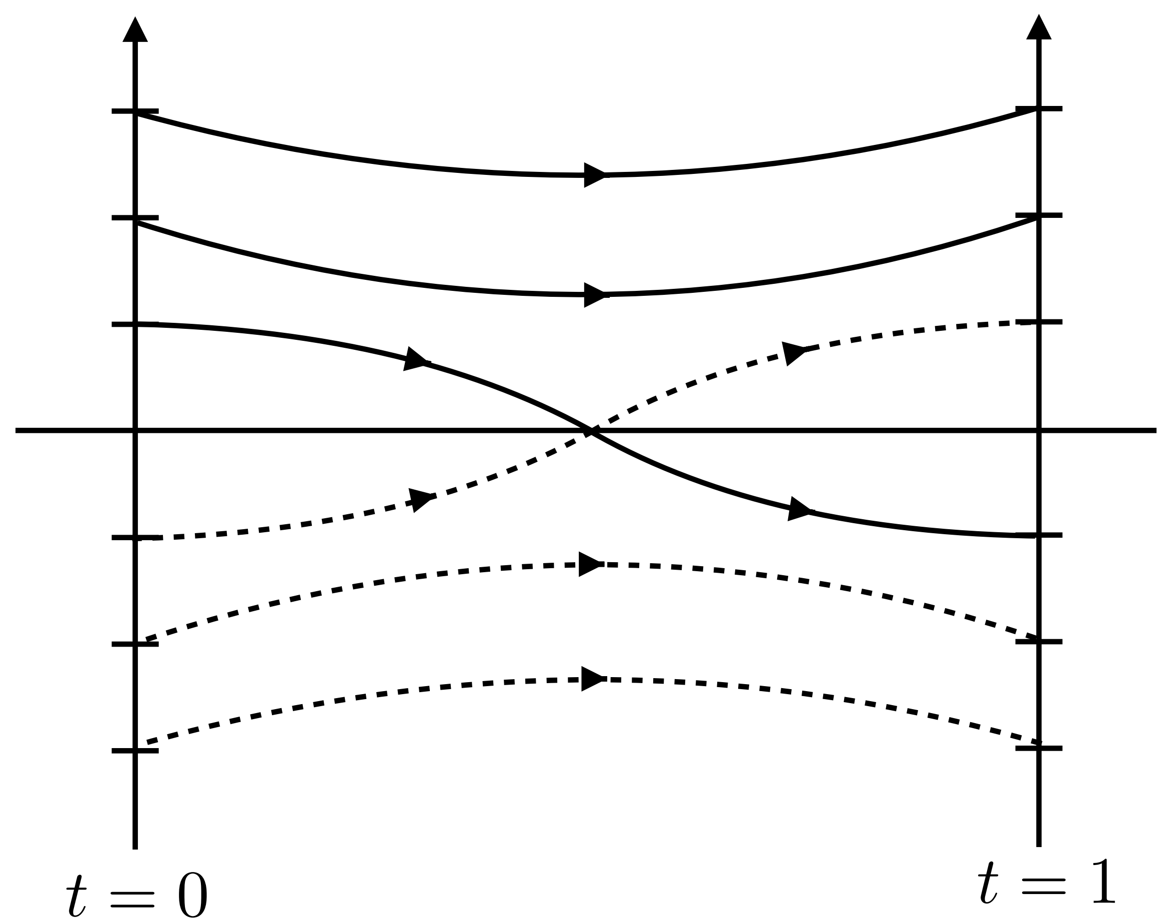
where U is a topologically nontrivial gauge transformation.

How do the positive eigenvalues "move" with t?

$$\mathscr{A}^t_\mu = (1-t)\mathscr{A}_\mu + t\mathscr{A}^U_\mu \qquad\qquad 0 \leq t \leq 1$$

Varying t induces a **spectral flow** of eigenvalues in which some may change sign:

$$[\det i\not{D}(\mathscr{A})]^{\frac{1}{2}} = \prod_{\lambda_n>0} \lambda_n$$



$$[\det i\not{D}(\mathscr{A}^U)]^{\frac{1}{2}} = \prod_{\lambda_n>0} \lambda^U_n$$

If an **odd number** of positive eigenvalues change sign, then

$$[\det i\not{D}(\mathscr{A}^U)]^{\frac{1}{2}} = -[\det i\not{D}(\mathscr{A})]^{\frac{1}{2}}$$

$$[\det i\not{D}(\mathscr{A}^U)]^{\frac{1}{2}} = -[\det i\not{D}(\mathscr{A})]^{\frac{1}{2}}$$

This would be a disaster, since after integrating over **all gauge fields**, the correlation functions of any gauge invariant operators vanish!

$$\int \mathscr{D}\mathscr{A}_\mu \det[i\not{D}(\mathscr{A})]^{\frac{1}{2}} \mathscr{O}_1 \ldots \mathscr{O}_n e^{-\frac{1}{4}\int d^4x \mathrm{Tr}\, \mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}} = 0$$

and the theory become "empty".

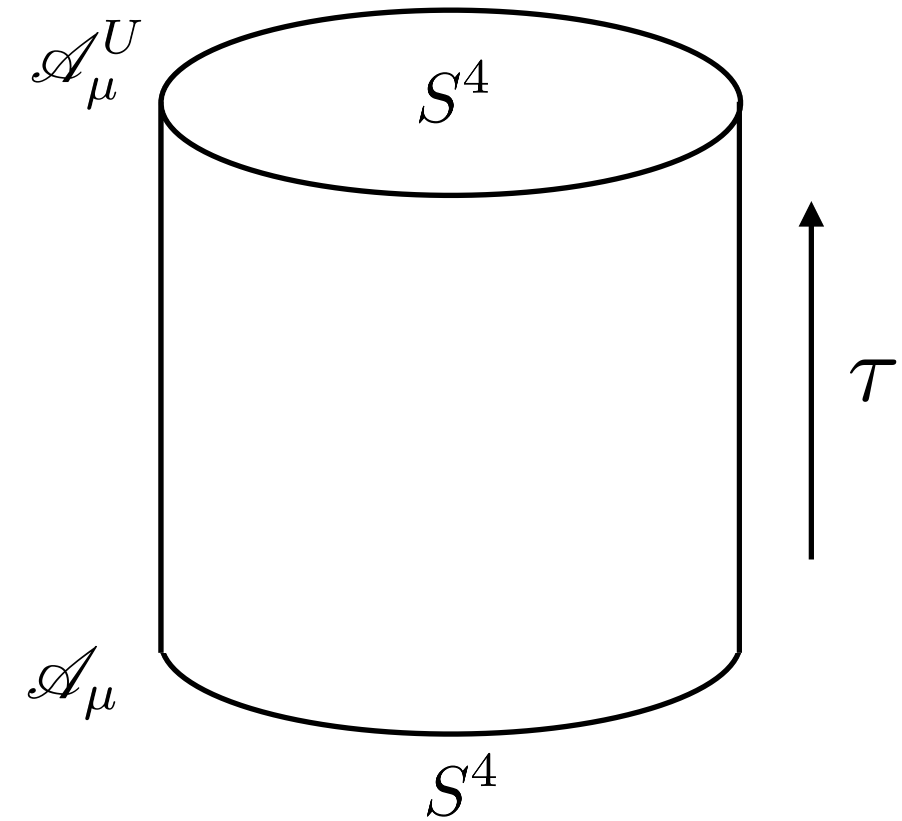
If the theory contains n $\mathrm{SU}(2)$ doublets, then the result of the fermionic integration is

$$\det[i\not{D}(\mathscr{A})]^{\frac{n}{2}}$$

and the conclusion is avoided if n is **even**.

But, is there really a **global** $\mathrm{SU}(2)$ **anomaly**?

We study a **five-dimensional** problem on the cylinder $S^4 \times \mathbb{R}$ with an **instanton-like** configuration…



… and define the Dirac operator

$$\not{D}^{(5)} = \gamma^\tau \frac{\partial}{\partial \tau} + \not{D}$$

The zero-mode equation is

$$\not{D}^{(5)} \psi = 0 \quad \Longrightarrow \quad \frac{\partial \psi}{\partial \tau} = -\gamma^\tau \not{D} \psi$$

The operators $\not{D}$ and $\gamma^\tau \not{D}$ have the **same spectrum**:

$$\not{D} \psi_n = \lambda_n \psi_n \quad \Longrightarrow \quad \gamma^\tau \not{D} (\mathbb{I} - \gamma^\tau) \psi_n = -\lambda_n (\mathbb{I} - \gamma^\tau) \psi_n$$

where $\{\gamma^\tau, \gamma^\mu\} = 0$ and $(\gamma^\tau)^2 = -\mathbb{I}$

$$\frac{\partial \psi}{\partial \tau} = -\gamma^\tau \not{D}\psi$$

Now, we assume that gauge field $\mathscr{A}_\mu(x,\tau)$ varies **adiabatically** with respect to τ

$$\psi(x,\tau) = F(\tau)\psi_\tau(x) \qquad \text{where} \qquad \gamma^\tau \not{D}\psi_\tau(x) = \lambda(\tau)\psi_\tau(x)$$

In the adiabatic approximation, the zero-mode equation $\not{D}^{(5)}\psi = 0$ reads

$$\frac{\partial \psi}{\partial \tau} = -\gamma^\tau \not{D}\psi \quad \Longrightarrow \quad F'(\tau) = -\lambda(\tau)F(\tau)$$

$$\Downarrow$$

$$F(\tau) = F(0)\exp\left[-\int_0^\tau dt'\lambda(t')\right]$$

and the zero-modes of the five-dimensional Dirac operator are

$$\psi(x,\tau) = F(0)\psi_\tau(x)\exp\left[-\int_0^\tau dt'\lambda(t')\right]$$

$$\psi(x,\tau) = F(0)\psi_\tau(x)\exp\left[-\int_0^\tau dt'\lambda(t')\right]$$

This mode is normalizable only if:

$\lambda(\tau) > 0$ for $\tau \longrightarrow \infty$

$\lambda(\tau) < 0$ for $\tau \longrightarrow -\infty$

With this adiabatic argument, we have shown that:

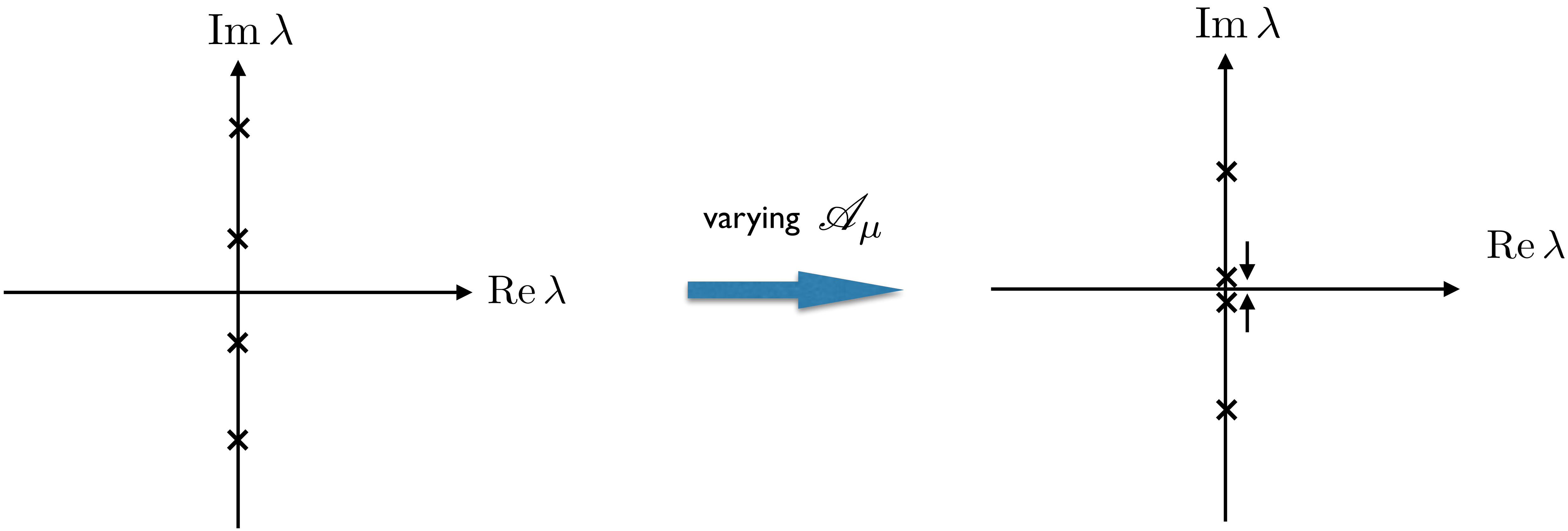
- The **normalizable** zero-modes of $\not{D}^{(5)}$ are in **one-to-one correspondence** with the eigenvectors of $\not{D}(\mathscr{A})$ **changing sign** with the spectral flow.

The question now is:

How many zero-modes does $\not{D}^{(5)}$ have?

mod 2 Atiyah-Singer index theorem

The operator $\not{D}^{(5)}$ is real and antisymmetric. Its eigenvalues are either **zero** or **purely imaginary** and come in complex conjugate **pairs**.



The number or zero-modes changes with a pair of **complex conjugate** eigenvalues moves towards or away the real axis.

The number of zero-modes of $\not{D}^{(5)}$ **mod 2** is a topological invariant.

The number (mod 2) of zero-modes of $\not{D}^{(5)}$ can be computed using the **mod 2 Atiyah-Singer index theorem** for the gauge instanton-like configuration:

$$\text{\# of zero-modes of } \not{D}^{(5)} = 1 \pmod 2$$

Thus, there is an **odd number** of eigenvalues of $\not{D}(\mathscr{A})$ changing sign as we deform the connection from $\mathscr{A}_\mu$ to $\mathscr{A}_\mu^U$

$$[\det i\not{D}(\mathscr{A}^U)]^{\frac{1}{2}} = -[\det i\not{D}(\mathscr{A})]^{\frac{1}{2}}$$

A theory with an **odd number** of **chiral** fermions transforming as **doublets** of $\mathrm{SU}(2)$ is anomalous!

Fortunately, the **standard model** is **safe**!

$$\left.\begin{matrix} \begin{pmatrix} e \\ \nu_e \end{pmatrix}_L & \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}_L & \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}_L \\ \\ \begin{pmatrix} u \\ d \end{pmatrix}_L & \begin{pmatrix} c \\ s \end{pmatrix}_L & \begin{pmatrix} t \\ b \end{pmatrix}_L \end{matrix}\right\} \quad 6\ \mathrm{SU}(2)_L \text{ doublets}$$

Both **leptons** and **quarks** are required to cancel the anomaly!

The **MSSM** is also **safe** due to the second Higgsino doublet

$$\left.\begin{matrix} \begin{pmatrix} \widetilde{h}_1^0 \\ \widetilde{h}_1^- \end{pmatrix}_L \\ \\ \begin{pmatrix} \widetilde{h}_2^+ \\ \widetilde{h}_2^0 \end{pmatrix}_L \end{matrix}\right\} \quad +\ 2\ \mathrm{SU}(2)_L \text{ doublets}$$

An index theorem computation of the gauge anomaly

We have managed to reformulate the problem of computing the axial anomaly into the calculation of the index of the Dirac-Weyl operator $D_+ \equiv \not{D}(\mathscr{A})P_+$

Let us try to do the same for the **gauge anomaly** in the simplest case of a chiral theory with a single **right-handed Weyl spinor** with gauge group G

We begin by computing the one-loop fermionic effective action in Euclidean space

$$e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[-\int d^4x \, \overline{\psi} i\not{D}(\mathscr{A})P_+\psi\right]$$

The gauge anomaly is given by the gauge variation of the effective action

$$\delta_\alpha \Gamma[\mathscr{A}] = -\int d^4x \, \alpha(x) D_\mu \langle J^\mu_R(x)\rangle_{\mathscr{A}}$$

To find the origin of the anomaly, let us consider the fermion in a **complex** representation R of the gauge group.

This theory is anomalous

$$\delta_\alpha \Gamma_R[\mathscr{A}] \neq 0$$

The same happens if the fermion is in the complex conjugate representation $\overline{R}$

$$\Gamma_{\overline{R}}[\mathscr{A}] = \Gamma_R[\mathscr{A}]^* \longrightarrow \delta_\alpha \Gamma_{\overline{R}}[\mathscr{A}] \neq 0$$

The theory with two fermions in the representations R and $\overline{R}$ is, however, anomaly free

$$\Gamma_{R\oplus\overline{R}}[\mathscr{A}] = \Gamma_R[\mathscr{A}] + \Gamma_{\overline{R}}[\mathscr{A}] \longrightarrow \delta_\alpha \Gamma_{R\oplus\overline{R}}[\mathscr{A}] = 0$$

Thus, **only the imaginary part** of the effective action is **anomalous**

$$\delta_\alpha\Big(\mathrm{Re}\,\Gamma_R[\mathscr{A}]\Big) = 0 \qquad \delta_\alpha\Big(\mathrm{Im}\,\Gamma_R[\mathscr{A}]\Big) \neq 0$$

$$e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[-\int d^4x\, \overline{\psi} i \not{D}(\mathscr{A}) P_+ \psi\right]$$

Being naive, we would just write

$$\Gamma[\mathscr{A}] = \log\det D_+ \qquad\qquad (D_+ = \not{D} P_+)$$

The problem is that **this determinant does not exist**…

The identity

$$\gamma_5 D_+ \equiv \gamma_5 \not{D}(\mathscr{A}) P_+ = -\not{D}(\mathscr{A})\gamma_5 P_+ = -\not{D}(\mathscr{A}) P_+ = -D_+$$

shows that D_+ maps positive chirality into negative chirality spinors

$$D_+ : S_+ \otimes E \longrightarrow S_- \otimes E \qquad\qquad E = \text{gauge bundle}$$

Since it is not an endomorphism, there is no eigenvalue problem and the **determinant cannot be defined**.

Instead, we work with a different **operator**

$$\widehat{D} : \Big(S_+ \oplus S_-\Big) \otimes E \longrightarrow \Big(S_+ \oplus S_-\Big) \otimes E$$

where

$$\widehat{D} = \left(\begin{array}{cc} 0 & \partial\!\!\!/_- \\ D_+ & 0 \end{array}\right) \qquad (\, \partial\!\!\!/_- \equiv \partial\!\!\!/ P_- \,)$$

This operator has a **well-defined eigenvalue problem** and the determinant can be computed.

This modification of the Weyl operator does not affect the anomaly, since **does not couple** to the **gauge field**

$$\boxed{D\!\!\!\!/ = \left(\begin{array}{cc} 0 & D_- \\ D_+ & 0 \end{array}\right)}$$

Its **modulus** is gauge invariant

$$|\det \widehat{D}|^2 = \det \widehat{D}\,\det \widehat{D}^\dagger = \det\Big(\partial\!\!\!/_+\partial\!\!\!/_-\Big)\det\Big(D_+D_-\Big) = \det\Big(\partial\!\!\!/_+\partial\!\!\!/_-\Big)\det D\!\!\!\!/$$

$$\Gamma[\mathscr{A}] = -\log\det\widehat{D}(\mathscr{A}) \qquad\qquad \operatorname{Re}\Gamma[\mathscr{A}] = -\log|\det\widehat{D}(\mathscr{A})|$$

Towards a topological interpretation of the gauge anomaly

(Álvarez-Gaumé & Ginsparg 1984)

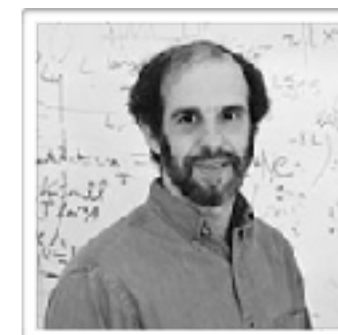
Let us **compactify** our $2n$-dimensional Euclidean space

$$\mathbb{R}^{2n} \cup \{\infty\} \longrightarrow S^{2n}$$

Luis Álvarez-Gaumé
(b. 1955)

and consider a **one-parameter family** of gauge transformations

$$g(x,\theta) \in G \qquad\qquad g(x,0) = g(x,2\pi) = \mathbb{I}$$



Paul Ginsparg
(b. 1955)

This defines a family of gauge transformations

$$\mathscr{A}^{\theta} = g(x,\theta)^{-1}(d + \mathscr{A})g(x,\theta)$$

where $\mathscr{A}$ is a **reference** connection such that $\not{D}(\mathscr{A})$ has **no zero modes**.

The transformation of $\det \widehat{D}(\mathscr{A})$ is

$$|\det \widehat{D}(\mathscr{A}^{\theta})| = |\det \widehat{D}(\mathscr{A})|$$

$$\det \widehat{D}(\mathscr{A}^{\theta}) = |\det \widehat{D}(\mathscr{A})| e^{iw(\theta,\mathscr{A})} = \sqrt{\det \not{D}(\mathscr{A})}\; e^{iw(\theta,\mathscr{A})}$$

$$\det \widehat{D}(\mathscr{A}^\theta) = \sqrt{\det \not{D}(\mathscr{A})}\; e^{iw(\theta,\mathscr{A})}$$

The **anomaly** is then given by the variation of the phase

$$\Gamma[\mathscr{A}] = -\log\det\widehat{D}(\mathscr{A}) \quad \longrightarrow \quad \delta_\alpha\Gamma[\mathscr{A}] = -i\delta\theta\frac{\partial}{\partial\theta}w(\theta,\mathscr{A})$$

The phase of the determinant defines a map

$$e^{iw(\theta,\mathscr{A})} : S^1 \longrightarrow S^1$$

classified by its **winding number**

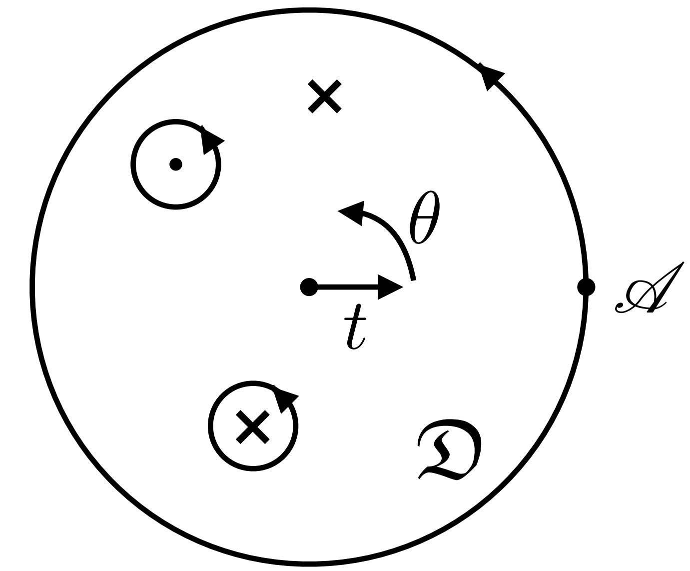
$$m = \frac{1}{2\pi}\int_0^{2\pi} d\theta\frac{\partial}{\partial\theta}w(\theta,\mathscr{A}) \in \mathbb{Z}$$

Thus, the anomaly is given by the winding number density!

The **gauge anomaly** admits a **topological interpretation**.

Is the winding number related with some kind of **index theorem**?

Let us consider the following connection defined on the manifold $S^{2n} \times \mathfrak{D}$



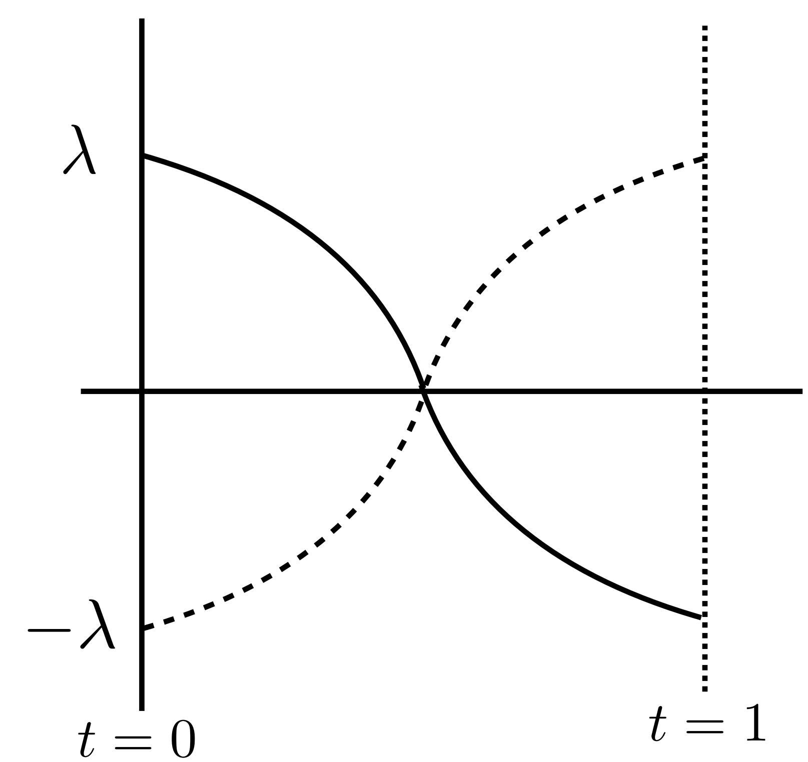
$$\mathscr{A}^{t,\theta}(x) = t\mathscr{A}^{\theta}(x) = g(x,\theta)^{-1}[d + \mathscr{A}(x)]g(x,\theta)$$

with $t \in [0,1]$.

By hypothesis, $\det \widehat{D}(\mathscr{A})$ does not vanish at $t = 1$. However, it may vanish at various points in the interior of $\mathfrak{D}$

$\mathscr{A}^{t,\theta}(x) = t\mathscr{A}^{\theta}(x)$ is not a gauge transformation!

The vanishing of $\det \widehat{D}(\mathscr{A})$ occurs when a pair of eigenvalues of the Dirac operator crosses zero.



We can define an extension of the Dirac operator to the interior of the disk. Introducing the $(D+2)$-dimensional gauge field

$$\mathfrak{a}_C(x,\theta,t) = (\mathscr{A}_\mu^{\theta,t}, 0, 0) \qquad\qquad C = 1,\ldots,D+2$$

the new Dirac operator takes the form

$$\not{D}(\mathfrak{a}) = \sum_{C=1}^{2n+2} \Gamma^C(\partial_C + \mathfrak{a}_C)$$

where

$$\Gamma^\mu = \sigma_1 \otimes \gamma^\mu$$
$$\Gamma^{2n+1} = \sigma_2 \otimes \gamma^\mu$$
$$\Gamma^{2n+2} = \sigma_1 \otimes \gamma_5$$

and

$$\Gamma_5 = \sigma_3 \otimes \mathbb{I}$$

It can be shown (long calculation) that the zero modes of $\not{D}_{2n+2}(\mathfrak{a})$ are in one-to-one correspondence with the zeroes of $\det \not{D}_{2n}(A^{\theta,t})$

The total winding number of the phase of $\det \not{D}_{2n}(A^{\theta,t})$ is the **sum of the winding numbers** of the vanishing eigenvalues.

$$m = \sum_i m_i$$

where

$$m_i = \pm 1$$

and moreover, m_i equals the chirality of the corresponding zero mode of the (2n+2)-dimensional Dirac operator $\not{D}_{2n+2}(\mathfrak{a})$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta} w(\theta, \mathscr{A}) = \mathrm{ind}\, \not{D}_{2n+2}(\mathfrak{a})$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta} w(\theta,\mathscr{A}) = \mathrm{ind}\, \not{D}_{2n+2}(\mathfrak{a})$$

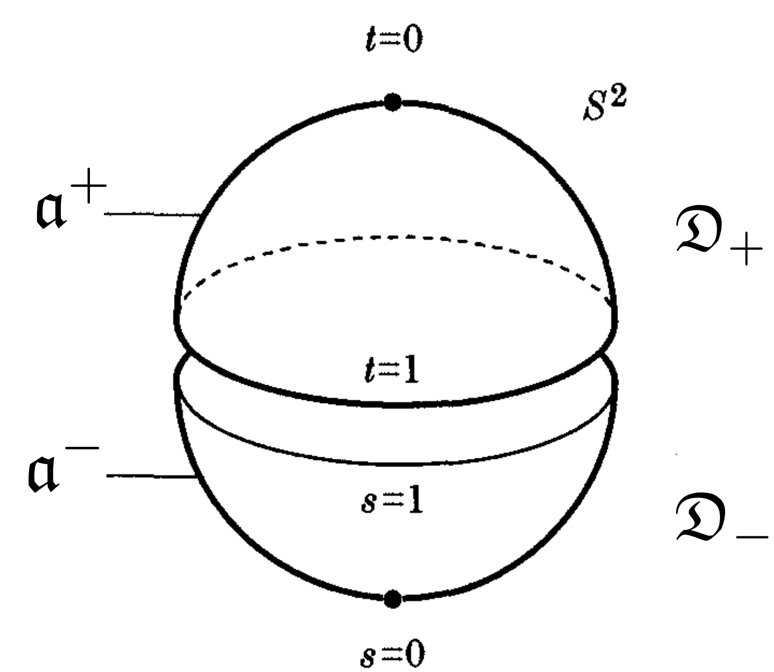
However, the Atiyah-Singer index theorem is **not applicable** because our manifold has a **boundary**!

We have two options:

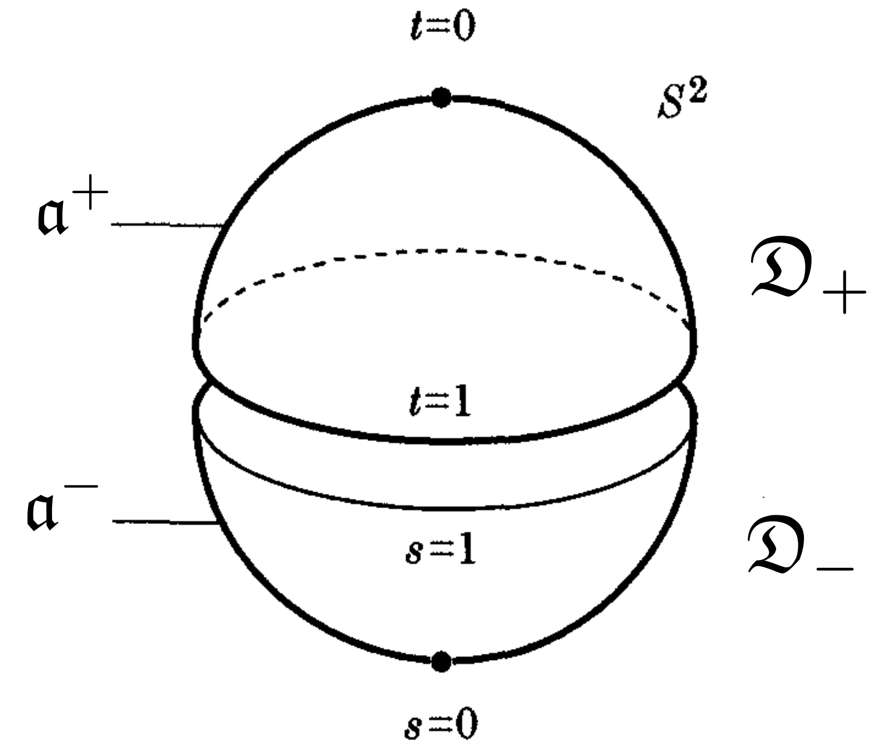
- Use the **Atiyah-Patodi-Singer index theorem** (valid for manifold with boundary).
- Set the boundary conditions by **gluing** two disks together to define the Dirac operator on the **closed manifold**

$$\underbrace{[S^{2n}\times\mathfrak{D}_+]\cup[S^{2n}\times\mathfrak{D}_-]}\longrightarrow S^{2n}\times S^2$$

nontrivial transition functions on $S^{2n}\times S^1$



Take the connection on the **upper hemisphere** to be ($d_\theta \equiv d\theta\partial_\theta$)



$$\mathfrak{a}^+(x,\theta,t) = tg(x,\theta)^{-1}d_\theta g(x,\theta) + \mathscr{A}^{t,\theta}(x)$$

while in the **lower hemisphere** we have

$$\mathfrak{a}^-(x,\theta,s) = \mathscr{A}(x)$$

At the hemisphere $t=s=1$, both connections are related by a **gauge transformation**

$$\mathfrak{a}^+(x,\theta,1) = g(x,\theta)^{-1}d_\theta g(x,\theta) + g(x,\theta)^{-1}dg(x,\theta) + g(x,\theta)^{-1}\mathfrak{a}^-(x)g(x,\theta)$$

The field strength is given by

$$\mathfrak{f}^+ = (d + d_\theta + d_t)\mathfrak{a}^+ + \mathfrak{a}^+ \wedge \mathfrak{a}^+$$

$$\mathfrak{f}^- = (d + d_\theta + d_t)\mathfrak{a}^- + \mathfrak{a}^- \wedge \mathfrak{a}^-$$

We can apply now the Atiyah-Singer index theorem to the Dirac operator in $S^{2n} \times S^2$

$$\text{ind}\, \not{D}_{2n+2}(\mathfrak{a}) = \int_{S^{2n}\times S^2} [\text{ch}(\mathfrak{f})]_{\text{vol}} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \int_{S^{2n}\times S^2} \text{Tr}\,\mathfrak{f}^{n+1}$$

The integral has to be computed as

$$\int_{S^{2n}\times S^2} \text{Tr}\,\mathfrak{f}^{m+1} = \int_{S^{2n}\times \mathfrak{D}_+} \text{Tr}\,(\mathfrak{f}^+)^{m+1} + \int_{S^{2n}\times \mathfrak{D}_-} \text{Tr}\,(\mathfrak{f}^-)^{m+1}$$

Locally, $\text{Tr}\,\mathfrak{f}^{n+1}$ is **exact** and on each hemisphere we have

$$\text{Tr}\,\mathfrak{f}^{n+1} = dQ_{2n+1}$$

and using Gauß' theorem, the integral gives in terms of the Chern-Simons form

$$\int_{S^{2n}\times S^2} \text{Tr}\,\mathfrak{f}^{m+1} = \int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^+,\mathfrak{f}^+)\Big|_{t=1} - \int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^-,\mathfrak{f}^-)\Big|_{s=1}$$

$$\mathrm{ind}\, \not{D}_{2n+2} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\left[\int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^+,\mathfrak{f}^+)\Big|_{t=1} - \int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^-,\mathfrak{f}^-)\Big|_{s=1}\right]$$

$$\mathfrak{a}^- = \mathscr{A}$$

$$\mathrm{ind}\, \not{D}_{2n+2} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\left[\int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^+,\mathfrak{f}^+)\Big|_{t=1} - \int_{S^{2n}\times S^1} Q_{2n+1}(\mathscr{A},\mathscr{F})\right]$$

We should recall that

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta} w(\theta,\mathscr{A}) = \mathrm{ind}\, \not{D}_{2n+2}$$

so we are only interested in those terms proportional to $d\theta$. Taking into account that

$$\mathfrak{a}^+ = \mathscr{A}^\theta + g^{-1}d_\theta g \equiv \mathscr{A}^\theta + \widehat{v} \qquad \mathfrak{f}^+(\mathscr{A}^\theta + \widehat{v}) = \mathfrak{f}^+(\mathscr{A}^\theta) \equiv \mathscr{F}^\theta$$

"Russian" formula

$$\mathrm{ind}\, \not{D}_{2n+2} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\int_{S^{2n}\times S^1} Q_{2n+1}(\mathscr{A}^\theta + \widehat{v},\mathscr{F}^\theta)$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta} w(\theta, \mathscr{A}) = \mathrm{ind}\, \not{D}_{2n+2}$$

$$\mathrm{ind}\, \not{D} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \int_{S^{2n}\times S^1} Q_{2n+1}(\mathscr{A}^\theta + \widehat{v}, \mathscr{F}^\theta)$$

From here we conclude

$$i d_\theta w(\theta, \mathscr{A}) = \frac{i^{n+2}}{(2\pi)^n (n+1)!}\int_{S^{2n}} Q^1_{2n+1}(\mathscr{A}^\theta + \widehat{v}, \mathscr{F}^\theta)$$

where $Q^1_{2n+1}(\mathscr{A}^\theta + \widehat{v}, \mathscr{F}^\theta)$ is the part of the Chern-Simons form linear in $\widehat{v}$

At the end of the calculation we can set $\theta = 0$ and $g(0,x) = \mathbb{I}$.

Thus, the **gauge anomaly** in D=$2n$ can be recast as the **axial anomaly** for a Dirac operator in D=$2n$+2

We have reached the **end of the course**…

There are a number of things **we did not have time** to discuss. For example:

- **Covariant vs. consistent** anomalies.
- **Wess-Zumino** terms.
- **Gravitational anomalies** in $D = 2, 6,$ and $10.$

$$\Downarrow$$

Green-Schwarz cancellation mechanism

- Other **advanced topics** (parity anomaly, anomalies on the lattice, anomaly inflow, etc.)

Thank you