Introduction to Anomalies in QFT

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Plan of the course

- Anomalies: general aspects
- The axial anomaly: a case study.
- Gauge anomalies
- Gravitational anomalies
- Anomalies and phenomenology:
  - Pion decay
  - Anomaly cancellation and model building
  - Nonperturbative physics from anomalies. Anomaly matching
- Functional methods
- Anomalies and topology
- Advanced topics (Green-Schwarz mechanism, anomaly inflow…)

Bibliography (a sample)

Books:

* R.A. Bertlmann, "Anomalies in Quantum Field Theory", Oxford 1996
* K. Fujikawa & H. Suzuki, "Path integrals and Quantum Anomalies", Oxford 2004
* L. Álvarez-Gaumé & M.A. Vázquez-Mozo, "Introduction to Anomalies", Springer (to appear)

General QFT books:

* M.E. Peskin & D.V. Schroeder, "An Introduction to Quantum Field Theory", Perseus Books 1995 (Chapter 19)
* L. Álvarez-Gaumé & M.A. Vázquez-Mozo, "An Invitation to Quantum Field Theory", Springer 2012 (Chapter 9)
* M.D. Schwartz, "Quantum Field Theory and the Standard Model", Cambridge 2014

Online Reviews:

* J.A. Harvey, "TASI Lectures on Anomalies", hep-th/0509097

What is an anomaly?

In certain situations, some symmetries/invariances of the classical theory can be incompatible with the quantization procedure

In those cases we say the theory has an **ANOMALY**, or that the symmetry/ invariance is **ANOMALOUS.**

The obvious example is **scale invariance**. E.g.

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4!}\phi^4 \right)$$

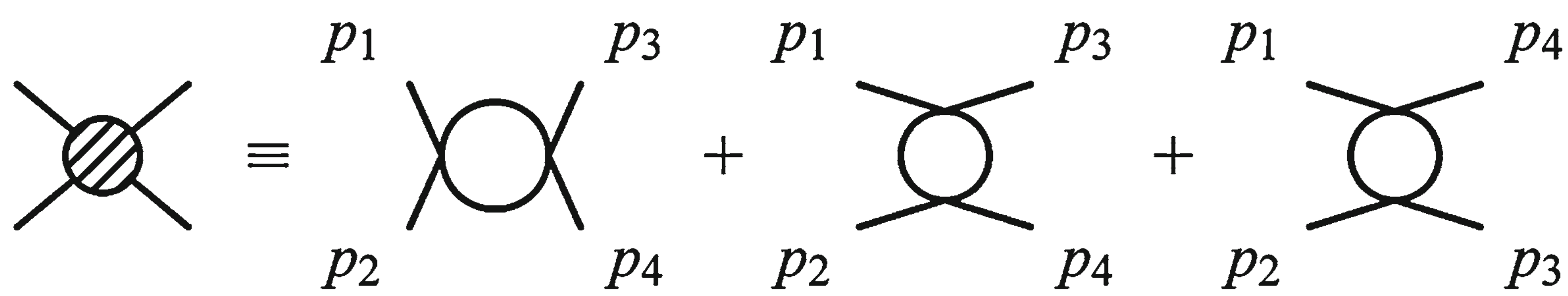
Is classically invariant under scale transformations

$$x^\mu \to \xi x^\mu,$$

$$\phi(x) \to \xi^{-1}\phi(\xi^{-1}x).$$

The physics is the same at all scales.

Upon quantization, however, we have divergences to deal with. For example,



The regularization of the integrals introduces an energy scale that leads to a running of the coupling:

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3}{16\pi^3}\lambda(\mu_0)\log\left(\frac{\mu}{\mu_0}\right)}$$

This quantum breaking of scale invariance is encoded in the beta function

$$\beta(\lambda) = \frac{3\hbar\lambda^2}{16\pi^2}$$

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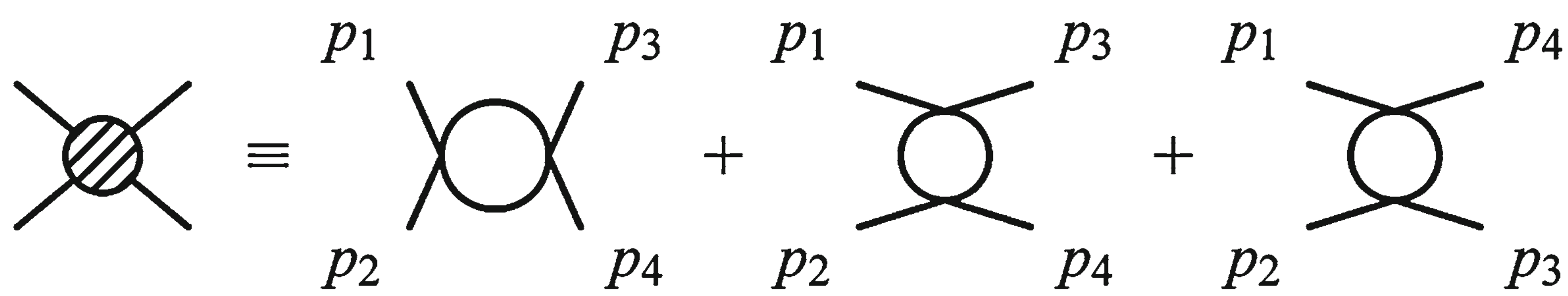

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Scale anomaly: a quantum mechanical toy model

We illustrate scale anomaly with a quantum mechanical example:

Let us take the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2M} + V(\mathbf{r})$$

where the potential is a homogeneous function of degree -2

$$V(\lambda\mathbf{r}) = \lambda^{-2}V(\mathbf{r})$$

The EOM

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}}$$

are invariant under

$$\begin{aligned} t &\longrightarrow \lambda^2 t, \\ \mathbf{r} &\longrightarrow \lambda\mathbf{r}, \\ \mathbf{p} &\longrightarrow \lambda^{-1}\mathbf{p}. \end{aligned}$$

In particular we consider the 2D potential:

$$V(\mathbf{r}) = \alpha\,\delta^{(2)}(\mathbf{r})$$

remember:

$$\delta^{(2)}(\lambda\mathbf{r}) = \frac{1}{\lambda^2}\delta^{(2)}(\mathbf{r})$$

The Schrödinger equation reads

$$-\frac{\hbar^2}{2M}\nabla^2\psi(\mathbf{r}) + \alpha\,\delta^{(2)}(\mathbf{r})\psi(\mathbf{0}) = \frac{\hbar^2\mathbf{k}^2}{2M}\psi(\mathbf{r}) \qquad (\,\hbar^2\mathbf{k}^2 = 2ME\,)$$

that we solve in momentum space:

$$\frac{1}{2M}(\mathbf{p}^2 - \mathbf{k}^2)\psi(\mathbf{p}) = -\alpha\psi(\mathbf{r}=\mathbf{0})$$

$$\Downarrow$$

$$\psi^{(\pm)}(\mathbf{p}) = (2\pi)^2\delta^{(2)}(\mathbf{p}\mp\mathbf{k}) - 2M\alpha\psi^{(\pm)}(\mathbf{r}=\mathbf{0})\frac{1}{\mathbf{p^2}-\mathbf{k^2}\mp i\epsilon}$$

We transform back to position space

$$\psi^{(+)}(\mathbf{r}) = e^{-i\mathbf{k}\cdot\mathbf{r}} - 2M\alpha\psi^{(\pm)}(\mathbf{0})\int\frac{d^2\mathbf{p}}{(2\pi)^2}\frac{e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}}{\mathbf{p}^2-\mathbf{k}^2-i\epsilon}$$

To obtain the spectrum of the theory, we set $\mathbf{r}=\mathbf{0}$ to obtain the **consistency condition**

$$\psi^{(+)}(\mathbf{0}) = 1 - 2M\alpha\psi^{(\pm)}(\mathbf{0})\int\frac{d^2\mathbf{p}}{(2\pi)^2}\frac{1}{\mathbf{p}^2-\mathbf{k}^2-i\epsilon}$$

However, the integral is **divergent**. Using a hard momentum cutoff

$$\int\frac{d^2\mathbf{p}}{(2\pi)^2}\frac{1}{\mathbf{p}^2-\mathbf{k}^2-i\epsilon} = \frac{1}{2\pi}\int_0^\Lambda\frac{pdp}{p^2-k^2-i\epsilon} = \frac{1}{4\pi}\log\left(-\frac{\Lambda^2}{\mathbf{k}^2}\right)$$

we have

$$\psi^{(+)}(\mathbf{0}) = \frac{1}{1+\frac{2M\alpha}{4\pi}\log\left(-\frac{\Lambda^2\hbar^2}{2ME}\right)}$$

With this we have a **solution** of the Schrödinger equation

$$\psi^{(+)}(\mathbf{r}) = e^{-i\mathbf{k}\cdot\mathbf{r}/\hbar} - \frac{1}{\frac{1}{2M\alpha} + \frac{1}{4\pi}\log\left(-\frac{\Lambda^2\hbar^2}{2ME}\right)} \int \frac{d^2\mathbf{p}}{(2\pi)^2} \frac{e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}}{\mathbf{p}^2 - \mathbf{k}^2 - i\epsilon}$$

The integral can be solved in terms of **Hankel functions** as

$$\int \frac{d^2\mathbf{p}}{(2\pi)^2} \frac{e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}}{\mathbf{p}^2 - \mathbf{k}^2 - i\epsilon} = \frac{i}{4} H_0^{(1)}\left(\frac{kr}{\hbar}\right)$$

Using its **asymptotic expansion** for large arguments

$$H_0^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{iz + \frac{i\pi}{4}}$$

we have (as $|\mathbf{r}| \to \infty$)

$$\psi^{(+)}(\mathbf{r}) \sim e^{-i\mathbf{k}\cdot\mathbf{r}/\hbar} - \frac{1}{\sqrt{2\pi kr}} \frac{e^{i\frac{kr}{\hbar} + \frac{i\pi}{4}}}{\frac{1}{M\alpha} + \frac{1}{2\pi}\log\left(-\frac{\Lambda^2\hbar^2}{2ME}\right)}.$$

Let us now **compare** our result

$$\psi^{(+)}(\mathbf{r}) \sim e^{-i\mathbf{k}\cdot\mathbf{r}/\hbar} - \frac{1}{\sqrt{2\pi k r}} \frac{e^{i\frac{kr}{\hbar}+\frac{i\pi}{4}}}{\frac{1}{M\alpha} + \frac{1}{2\pi}\log\left(-\frac{\Lambda^2\hbar^2}{2ME}\right)}$$

with the asymptotic for of the wave function for 2D scattering

$$\psi^{(+)}(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}/\hbar} + \frac{e^{i\frac{kr}{\hbar}+\frac{i\pi}{4}}}{\sqrt{r}} f(\theta)$$

We identify the scattering function as

$$f(\theta) = -\frac{1}{\sqrt{2\pi k}} \frac{1}{\frac{1}{M\alpha} + \frac{1}{2\pi}\log\left(-\frac{\Lambda^2\hbar^2}{2ME}\right)}$$

Two important **features**:

- It is **independent of the angle** (only s-wave scattering).
- It **depends** on the (unphysical) **cutoff**.

To deal with the **second problem**, we notice that the scattering function

$$f(\theta) = -\frac{1}{\sqrt{2\pi k}} \frac{1}{\frac{1}{M\alpha} + \frac{1}{2\pi}\log\left(-\frac{\Lambda^2\hbar^2}{2ME}\right)}$$

has a **pole** for negative energy

$$E_0 = -\frac{\Lambda^2\hbar^2}{2M} e^{\frac{2\pi}{M\alpha}} < 0.$$

[at this point, sending $\Lambda \to \infty$ requires $\alpha \to 0^-$]

Thus, the theory has a **single bound state** and we can trade the cutoff Λ by the observable quantity E_0

$$f(\theta) = \sqrt{\frac{2\pi}{k}} \frac{1}{\log\left(\frac{\hbar^2 k^2}{2M|E_0|}\right) - i\pi}$$

We have **renormalized** the theory!

We define the scattering function in terms of the **phase shifts**

$$f(\theta) = -i \sum_{n=-\infty}^{\infty} \frac{e^{2i\delta_n(k)} - 1}{\sqrt{2\pi k}} e^{in\theta}$$

Using our result for $f(\theta)$,

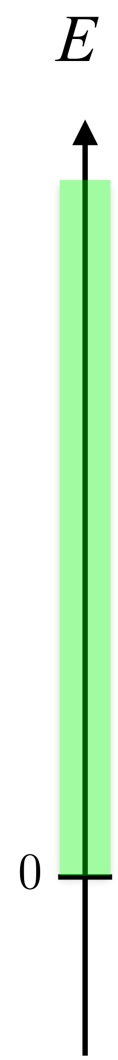
$$e^{2i\delta_0(k)} = \frac{\frac{1}{\pi}\log\left(\frac{\hbar^2 k^2}{2M|E_0|}\right) + i}{\frac{1}{\pi}\log\left(\frac{\hbar^2 k^2}{2M|E_0|}\right) - i} \qquad \Longrightarrow \qquad \delta_n(k) = \delta_{n,0}\cot^{-1}\left[\frac{1}{\pi}\log\left(\frac{\hbar^2 k^2}{2M|E_0|}\right)\right].$$

The phase shift depends on the **particle energy**, hence…

Scale invariance is broken!

What is going on here?

Classically, the spectrum is **scale invariant:**



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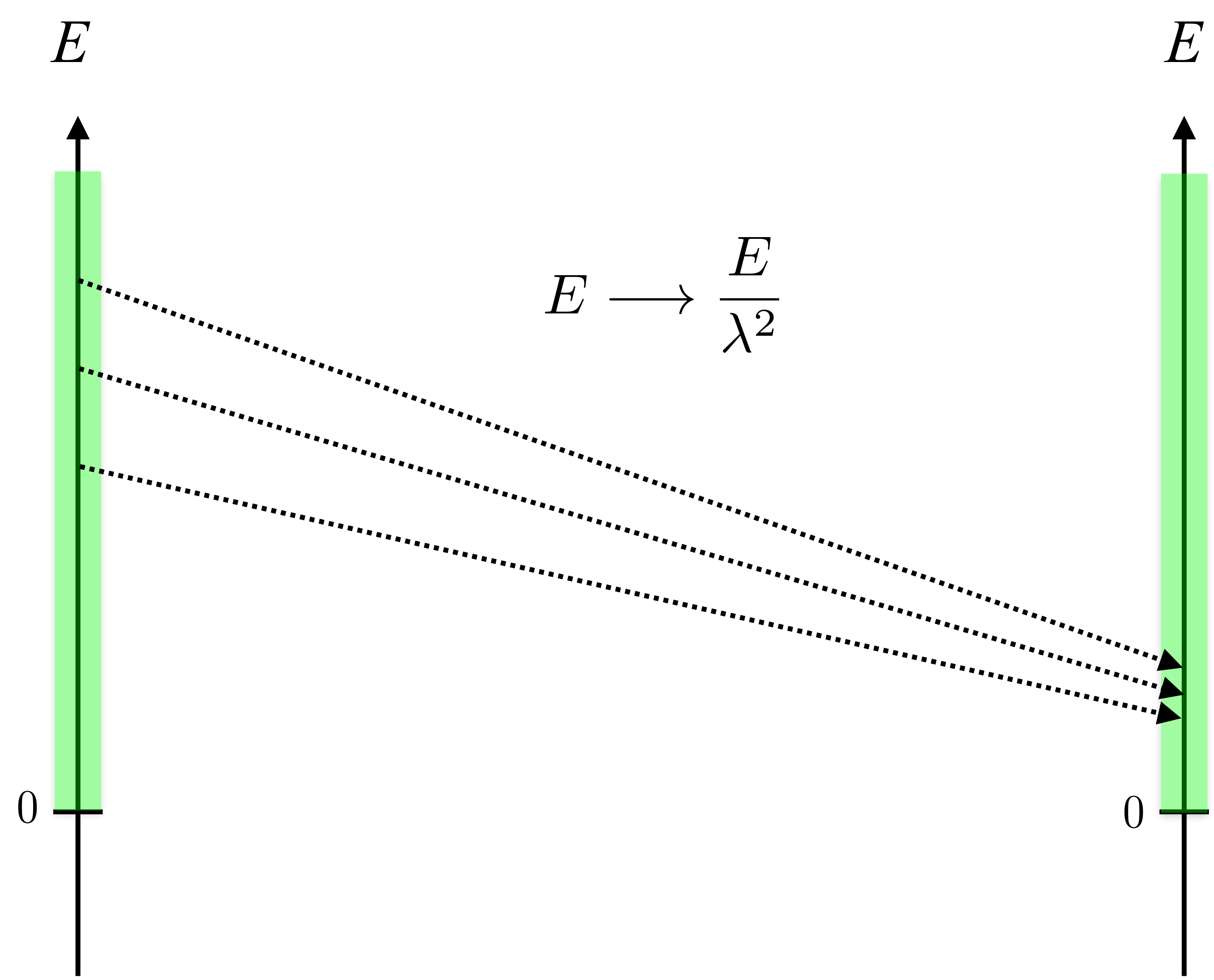
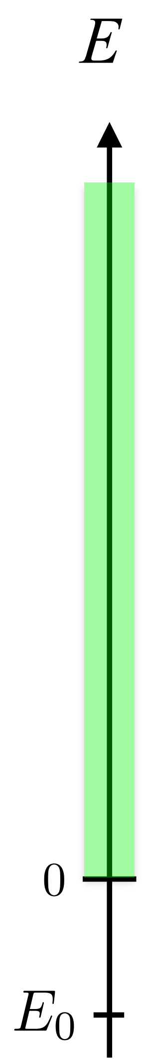


Quantum mechanically, **scale invariance is broken** by the presence of the **bound state:**



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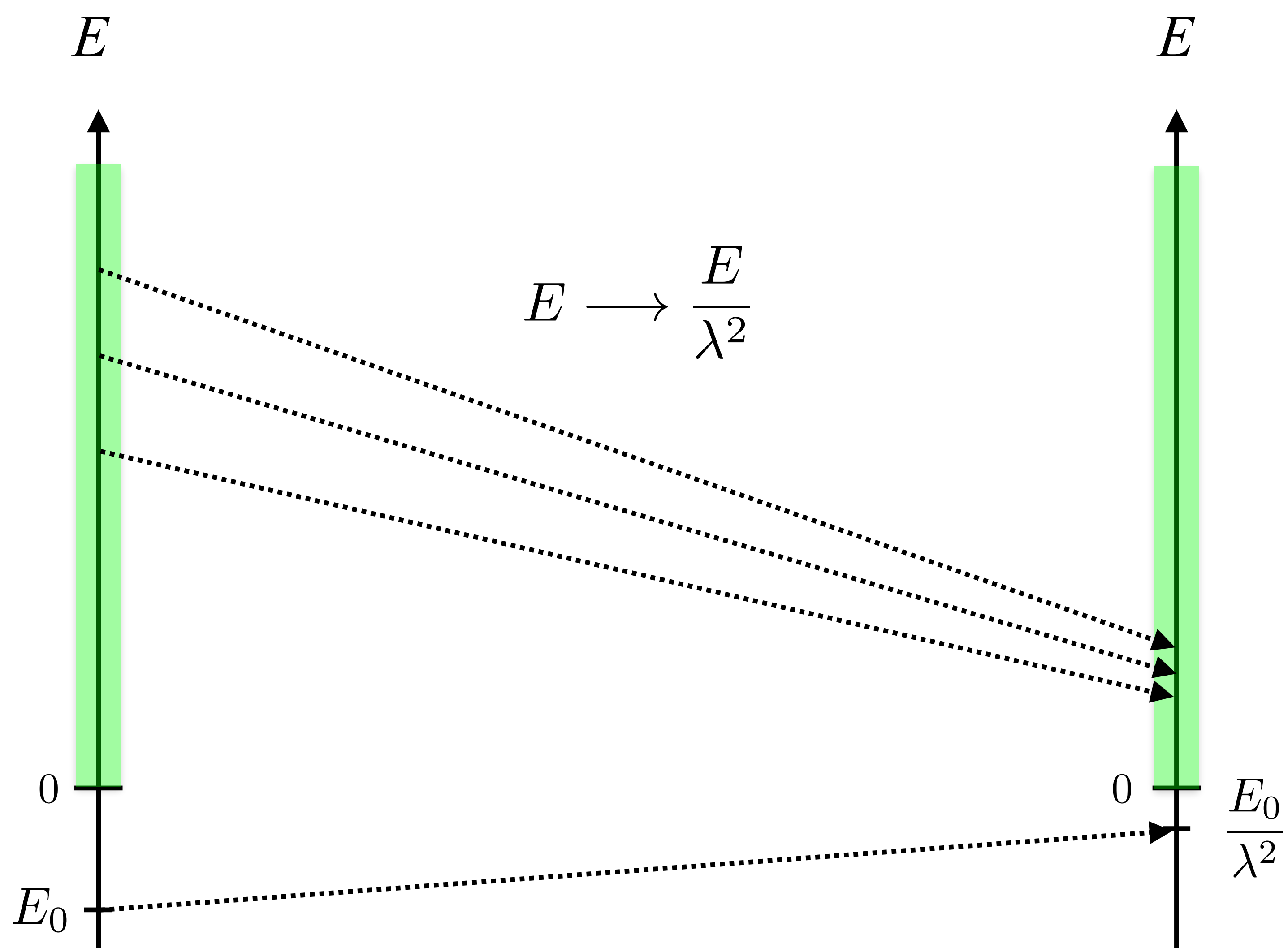

$$E_0 = -\frac{\Lambda^2\hbar^2}{2M}e^{\frac{2\pi}{M\alpha}} < 0.$$

We have an energy scale that is quantum-mechanically generated. (e.g. as in QCD)

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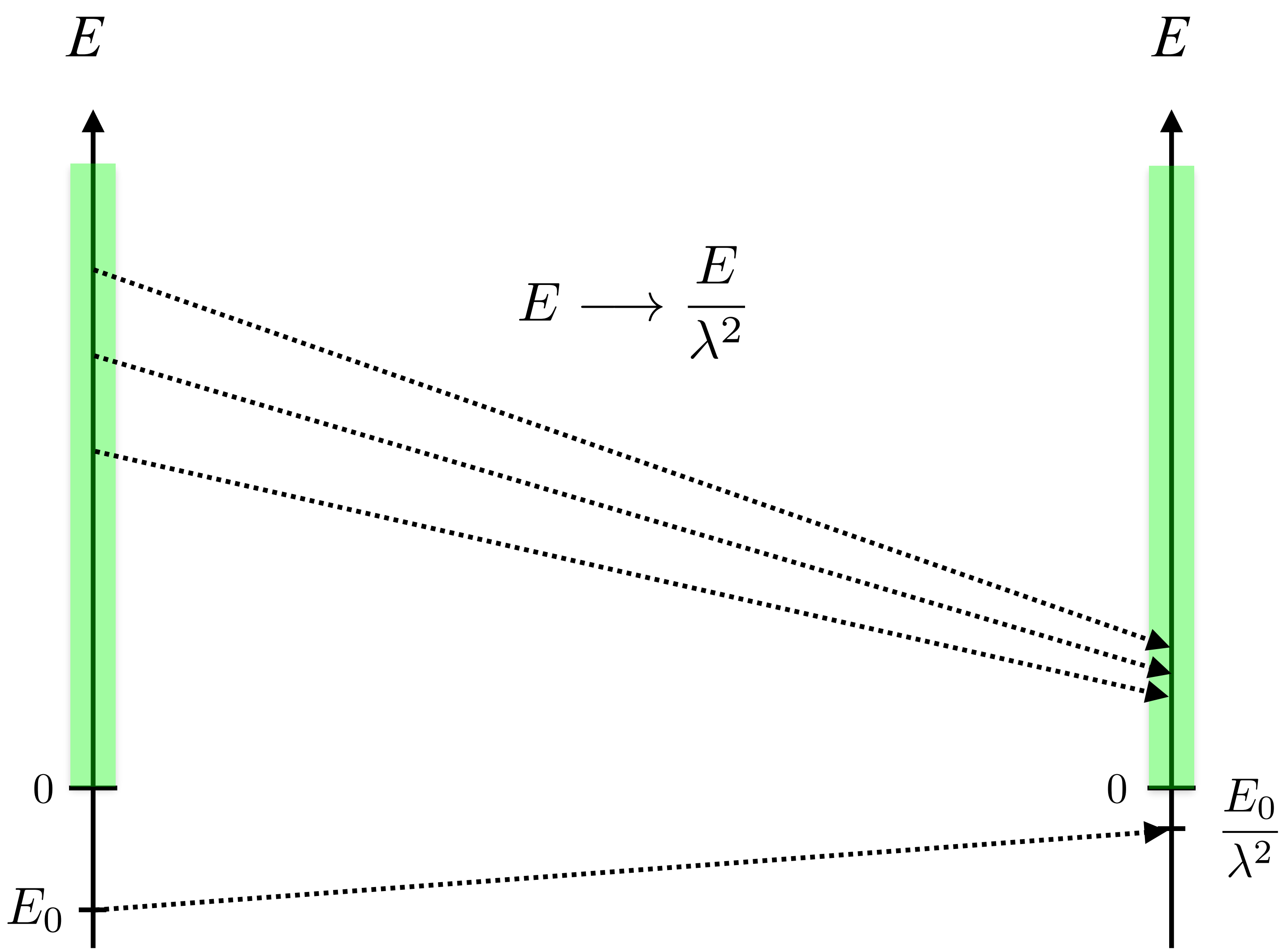


this bound state is a quantum mechanical affair!

$$E_0 = -\frac{\Lambda^2\hbar^2}{2M} e^{\frac{2\pi}{M\alpha}} < 0.$$

We have an energy scale that is quantum-mechanically generated. (e.g. as in QCD)

A **second example** is the 3D Hamiltonian

$$H = \frac{\mathbf{p}^2}{2M} + \frac{\alpha}{\mathbf{r}^2}$$

For the attractive case ($\alpha < 0$) the potential overcomes the centrifugal barrier for

$$\left(\ell + \frac{1}{2}\right)^2 < 3M|\alpha|$$

and the spectrum becomes continuous and unbounded from below

The Hamiltonian is not self-adjoint!

To define the theory we regularize the Hamiltonian near $\mathbf{r} = \mathbf{0}$, e.g.

$$V(\mathbf{r}) = \begin{cases} \frac{\alpha}{\mathbf{r}^2} & |\mathbf{r}| > a \\ \infty & |\mathbf{r}| < a \end{cases}$$

Renormalizing the parameters of the solution, leads in the $a \to 0$ again to a **bound state** and the **breaking of scale invariance**.

[see e.g. Coon & Holstein, Am. J. Phys. **70** (2002) 513]

The physics of these toy models is similar to **dimensional transmutation** in QCD

$$S_{\text{QCD}} = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_{f=1}^{N_f} \overline{Q}^f i\not{D}\, Q^f \right)$$

$$\Big\Downarrow \quad \text{quantization}$$

$$\Lambda_{\text{QCD}}$$

Anomalies: the good and the bad

Whether anomalies are bad or good depends on what symmetries/invariances they affect:

- They are **harmless and even useful** when they affect **global** (non gauge) symmetries

  - Scale invariance $\longrightarrow$ asymptotic freedom

  - Chiral symmetry $\longrightarrow$ $\pi^0 \longrightarrow 2\gamma$

Their presence can be also used to extract **nonpeturbative information** about the theory (anomaly matching)

- They are potentially disastrous when they affect **gauge symmetries**

  - Gauge anomalies

  - Gravitational anomalies

These types of anomalies **should be cancelled at all cost**, otherwise the theory becomes sick (e.g. nonunitary)

The conditions for anomaly cancellations can be useful for phenomenology (e.g. constraints on the spectrum)

The axial anomaly

The symmetries of QED: a reminder

The QED action

$$S_{\mathrm{QED}} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}A\!\!\!/\psi \right]$$

is invariant under **global** U(1)$_{\mathrm{V}}$ transformations of the fermion field

$$\psi(x) \longrightarrow e^{i\alpha}\psi(x), \qquad \overline{\psi}(x) \longrightarrow e^{-i\alpha}\overline{\psi}(x), \qquad \text{with} \qquad \alpha \in \mathbb{R}$$

leading to the **conservation equation**

$$J^{\mu}_{\mathrm{V}} = \overline{\psi}\gamma^{\mu}\psi \qquad \Longrightarrow \qquad \partial_{\mu} J^{\mu}_{\mathrm{V}} = 0$$

This symmetry can be promoted to U(1) **gauge invariance**

$$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x), \qquad A_{\mu}(x) \longrightarrow A_{\mu}(x) + \partial_{\mu}\alpha(x)$$

We can also allow a second type of **axial** global transformations of the fermion field:

$$\psi(x) \longrightarrow e^{i\beta\gamma_5}\psi(x), \quad \overline{\psi}(x) \longrightarrow \overline{\psi}(x)e^{i\beta\gamma_5}, \qquad \text{with} \qquad \beta \in \mathbb{R}$$

where

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

This is not a symmetry of the action, due to the **mass term**. If we define the **axial-vector current**

$$J^\mu_{\mathrm{A}} = \overline{\psi}\gamma^\mu\gamma_5\psi$$

it satisfies

$$\partial_\mu J^\mu_A = 2im\overline{\psi}\gamma_5\psi. \qquad \text{(pseudovector-pseudoscalar equivalence)}$$

Axial global symmetry is **recovered** in the massless limit $m \longrightarrow 0$

At the level of the scattering amplitudes, conservations equations give rise to **Ward identities**.

In the case of QED, a general amplitude in momentum space has the structure

$$\begin{aligned}\mathcal{A}(p_1,\ldots,p_n;q_1,\ldots,q_m) &= \varepsilon_{\mu_1}(p_1)\ldots\varepsilon_{\mu_n}(p_n)\varepsilon_{\nu_1}(q_1)^*\ldots\varepsilon_{\nu_n}(q_m)^* \\ &\times \Gamma^{\mu_1\ldots\mu_n\nu_1\ldots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m)\end{aligned}$$

Invariance under gauge transformations

$$\varepsilon_\mu(p)\longrightarrow\varepsilon_\mu(p)+\lambda p_\mu$$

leads to the **gauge Ward identity**

$$p_{\mu_i}\Gamma^{\ldots\mu_i\ldots\nu_1\ldots\nu_m}(p_k;q_\ell)=0=q_{\nu_i}\Gamma^{\mu_1\ldots\mu_m\ldots\nu_i\ldots}(p_k;q_\ell).$$

Or more generally, $\langle\partial_\mu J^\mu_{\rm V}(y)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle=0$ with $\mathcal{O}_i(x)$ gauge invariant operators.

What about the axial-vector current?

We study a Dirac fermion coupled to an **external gauge field** $\mathscr{A}_\mu(x)$

$$S_{\rm int} = -e\int d^4x\, J^\mu_{\rm V}(x)\mathscr{A}_\mu(x)$$

and compute

$$\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{\displaystyle\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, J^\mu_{\rm A}(x) e^{i\int d^4x\left[(i\partial\!\!\!/ - m)\psi - eJ^\mu_{\rm V}\mathscr{A}_\mu\right]}}{\displaystyle\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{i\int d^4x\left[\overline{\psi}(i\partial\!\!\!/ - m)\psi - eJ^\mu_{\rm V}\mathscr{A}_\mu\right]}}.$$

Expanding in **perturbation theory** in the coupling constant,

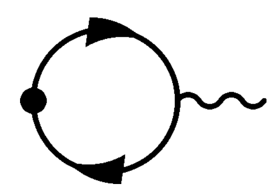
$$\begin{aligned}\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} &= -ie\int d^4y\,\langle 0|T[J^\mu_{\rm A}(x)J^\alpha_{\rm V}(y)]|0\rangle\mathscr{A}_\alpha(y)\\ &- \frac{e^2}{2}\int d^4y_1 d^4y_2\,\langle 0|T[J^\mu_{\rm A}(x)J^\alpha_{\rm V}(y_1)J^\beta_{\rm V}(y_2)]|0\rangle\mathscr{A}_\alpha(y_1)\mathscr{A}_\beta(y_2) + \ldots\end{aligned}$$

We look at the **first term**

$$e\langle 0|T[J^{\mu}_{\rm A}(0)J^{\alpha}_{\rm V}(y-x)]|0\rangle = \int \frac{d^4p}{(2\pi)^4}\Gamma^{\mu\alpha}(k)e^{ik\cdot(x-y)}$$

and **diagrammatically**:

$$i\Gamma^{\mu\nu}(k) \quad = $$



k

$$= \quad e\int \frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{i}{\not{\ell}-m+i\epsilon}\gamma^\nu\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}\right)$$

Our aim is to compute its contribution to the axial-vector Ward identity

$$\langle\partial_\mu J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \ ? \qquad \longrightarrow \qquad k_\mu i\Gamma^{\mu\nu}(k) = \ ?$$

To compute the integral

$$i\Gamma^{\mu\nu}(k) \;=\; e\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{i}{\not{\ell}-m+i\epsilon}\gamma^\nu\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}\right)$$

we use some **Diracology**

$$\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\right)=\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\right)=0,\quad \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right)=-4i\epsilon^{\mu\nu\alpha\beta}$$

to find

$$i\Gamma^{\mu\nu}(k) \;=\; -4ie\,\epsilon^{\mu\alpha\nu\beta}k_\beta\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell_\alpha}{(\ell^2-m^2+i\epsilon)[(\ell-k)^2-m^2+i\epsilon]}.$$

Due to the **antisymmetry** of $\epsilon_{\mu\nu\alpha\beta}$ the amplitude satisfy **both** the vector and axial-vector Ward identities

$$k_\mu i\Gamma^{\mu\nu}(k)=0=k_\nu i\Gamma^{\mu\nu}(k).$$

Moreover, by **Lorentz invariance** $i\Gamma^{\mu\nu}=0$

To find any anomaly, we have to go to the **next order**. Going to momentum space

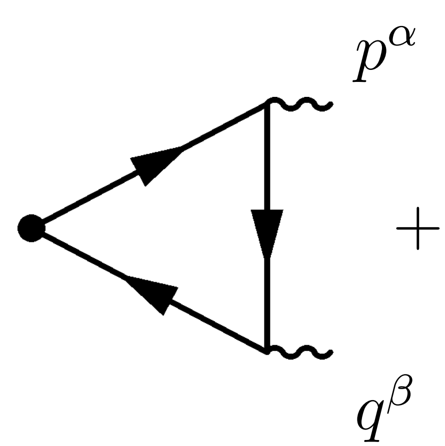
$$e^2\langle 0|T[J_{\rm A}^\mu(0)J_{\rm V}^\alpha(x_1)J_{\rm V}^\beta(x_2)]|0\rangle = \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}i\Gamma^{\mu\alpha\beta}(p,q)e^{ip\cdot x_1+iq\cdot x_2},$$

the conservation equation is

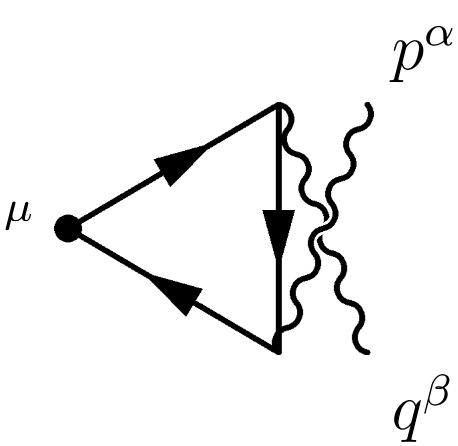
$$\begin{aligned}\partial_\mu\langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} &= \frac{i}{2}\int d^4y_1d^4y_2\mathscr{A}^\alpha(y_1)\mathscr{A}^\beta(y_2) \\ &\times \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q)e^{ip\cdot(y_1-x)+iq\cdot(y_2-x)}.\end{aligned}$$

The calculation involves now two **triangle diagrams**:

$i\Gamma_{\mu\alpha\beta}(p,q) = (p+q)^\mu$



$+\ (p+q)^\mu$



Applying the Feynman rules of QED, we have

$$i\Gamma_{\mu\alpha\beta}(p,q) = e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not{\ell}-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right)$$

$$+ \left(\begin{array}{c} p\leftrightarrow q \\ \alpha\leftrightarrow\beta\end{array}\right).$$

so we only need to compute the integrals…

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$$
\begin{aligned}
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&+ \left(\begin{array}{c} p \leftrightarrow q \\ \alpha \leftrightarrow \beta \end{array}\right).
\end{aligned}
$$

so we only need to compute the integrals...

These integrals are linearly divergent!!

Interlude: linearly divergent integrals

Let us begin with the simplest **one-dimensional case**:

$$I(\xi) = \int_{-\infty}^{\infty} dx \Big[f(x+\xi) - f(x) \Big].$$

If the function $f(x)$ is integrable on $\mathbb{R}$ we conclude that $I(\xi) = 0$.

Let us however assume that **for large** $|x|$ has one of the two behaviors:

$f(x) \sim \dfrac{1}{x}$ (logarithmically divergent integral)

$f(x) \sim \text{constant}$ (linearly divergent integral)

expanding the integrand around x

$$I(\xi) = \int_{-\infty}^{\infty} dx \left[f'(x)\xi + \frac{1}{2} f''(x)\xi^2 + \ldots \right]$$

we arrive at: $I(\xi) = \xi \displaystyle\int_{-\infty}^{\infty} dx\, f'(x) = \xi \Big[f(\infty) - f(-\infty) \Big].$

Thus, for linearly divergent integrals:

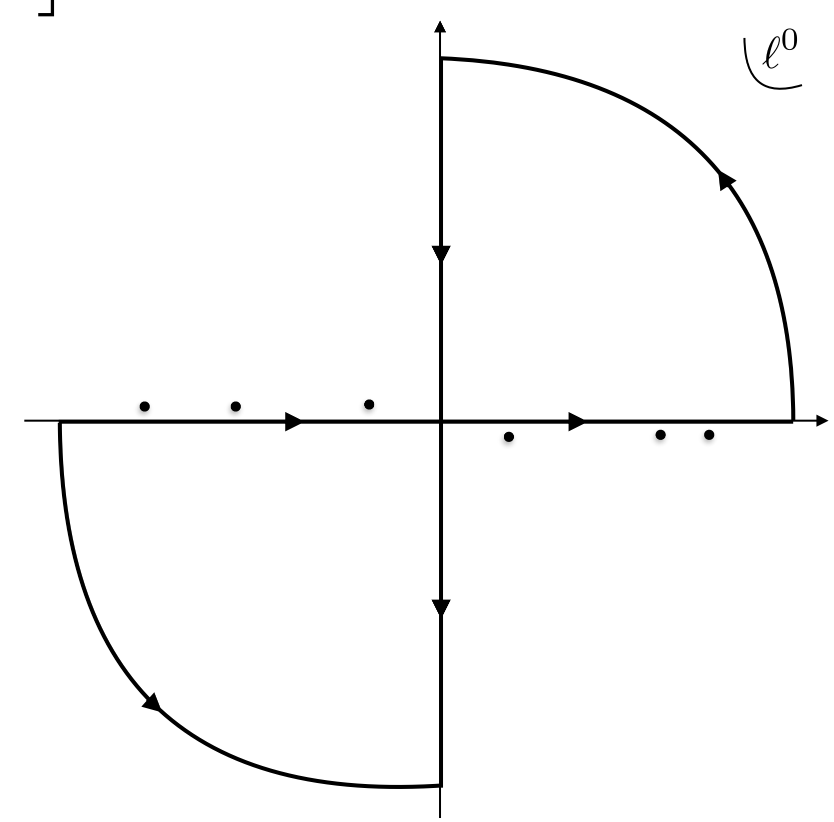
$$\int_{-\infty}^{\infty} dx \Big[f(x+\xi) - f(x) \Big] = f(\infty) - f(-\infty) \neq 0.$$

Shifting the integration variable **changes the value** of a linearly divergent integral!

Something similar happens in **four dimensions**

$$I_4^\mu(\xi) = \int \frac{d^4\ell}{(2\pi)^4} \Big[f^\mu(\ell+\xi) - f^\mu(\ell) \Big].$$

To make sense of the integral, we perform a **Wick rotation** into Euclidean space



$$I_4^\mu(\xi) = i \int \frac{d^4\ell_E}{(2\pi)^4} \Big[f^\mu(\ell_E+\xi) - f^\mu(\ell_E) \Big].$$

If the integral is linearly divergent its **asymptotic** behavior is:

$$f^\mu(\ell_E) \sim C\frac{\ell_E^\mu}{\ell_E^4} \qquad \textbf{as} \qquad |\ell_E| \longrightarrow \infty$$

Expanding the integrand

$$I_4^\mu(\xi) = i\int \frac{d^4\ell_E}{(2\pi)^4}\Big[f^\mu(\ell_E+\xi) - f^\mu(\ell_E)\Big]$$
$$= i\int \frac{d^4\ell_E}{(2\pi)^4}\left[\xi^\alpha \left.\frac{\partial f^\mu}{\partial \ell_E^\alpha}\right|_{\xi=0} + \frac{1}{2}\xi^\alpha\xi^\beta \left.\frac{\partial^2 f^\mu}{\partial \ell_E^\alpha \partial \ell_E^\beta}\right|_{\xi=0} + \ldots\right]$$

Again, only the first term contributes. Applying Gauß' theorem

$$I_4^\mu(\xi) \;=\; \frac{i}{16\pi^4}\int_{S^3_\infty} d\Sigma_\alpha \xi^\alpha f^\mu(\ell_E) = \frac{iC}{16\pi^4}\xi_\alpha \int d\Omega_3 \frac{\ell_E^\mu \ell_E^\alpha}{\ell_E^2}$$

The remaining integral can be done using **asymptotic rotational invariance**

$$\int d\Omega_3 \frac{\ell_E^\mu \ell_E^\alpha}{\ell_E^2} = \frac{1}{4}\delta^{\mu\alpha}\mathrm{Vol}(S^3) = \frac{\pi^2}{2}\delta^{\mu\alpha}$$

With this, we got

$$\int \frac{d^4\ell}{(2\pi)^4}\Big[f^\mu(\ell+\xi) - f^\mu(\ell)\Big] = \frac{iC}{32\pi^2}\xi^\mu.$$

Very important: remember the origin of the constant C

$$f^\mu(\ell_E) \sim C\frac{\ell_E^\mu}{\ell_E^4} \qquad \text{as} \qquad |\ell_E| \longrightarrow \infty$$

Thus, the ambiguity **only** depends on the **large momentum behavior** of the integrand (i.e., it doesn't depend on the masses of the particles running in the loop!)

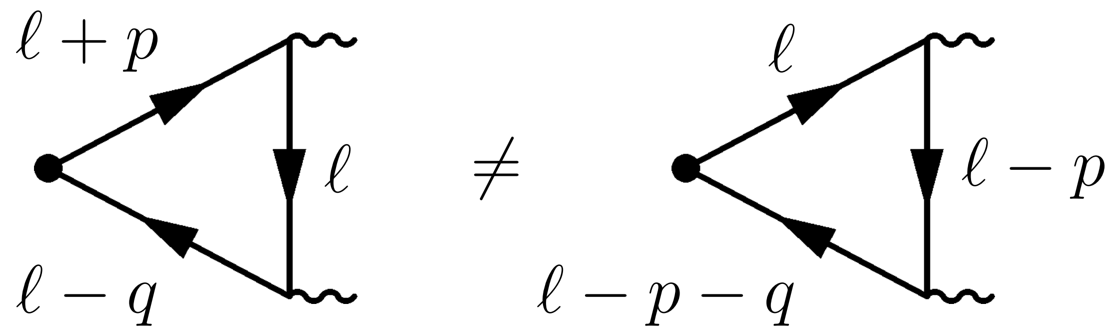
Back to the axial anomaly…

Remember that **applying the Feynman rules** of QED, we had obtained

$$i\Gamma_{\mu\alpha\beta}(p,q) = e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not{\ell}-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right)$$

$$+ \left(\begin{array}{c} p\leftrightarrow q \\ \alpha\leftrightarrow\beta\end{array}\right).$$

What is the relevance of the previous discussion?



The value of the triangle diagram depends on **how we parametrize** the loop momentum!

$$i\Gamma_{\mu\alpha\beta}(p,q) = e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not{\ell}-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right)$$

$$+ \left(\begin{array}{c} p\leftrightarrow q \\ \alpha\leftrightarrow\beta\end{array}\right).$$

We start with the **first term** in the computation of $(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q)$:

$$I_{\alpha\beta}(\ell,p,q) = \mathrm{Tr}\left[\frac{i}{\not{\ell}-m+i\epsilon}(\not{p}+\not{q})\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right].$$

To reduce the expression, we use the trivial identity

$$\not{p}+\not{q} = (\not{\ell}-m)-(\not{\ell}-\not{p}-\not{q}+m)+2m$$

and write

$$\frac{i}{\not{\ell}-m+i\epsilon}(\not{p}+\not{q})\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m} = i\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}+i\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5$$

$$+ 2m\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}$$

The integrand takes the form

$$\begin{aligned}
I_{\alpha\beta}(\ell,p,q) &= i\mathrm{Tr}\left(\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right) \\
&- i\mathrm{Tr}\left(\gamma_5\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\frac{i}{\not{\ell}-m+i\epsilon}\gamma_\beta\right) \\
&+ 2m\mathrm{Tr}\left(\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right)
\end{aligned}$$

and integrate the result over the loop momentum

$$(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) = e^2\int\frac{d^4\ell}{(2\pi)^4}I_{\alpha\beta}(\ell,p,q) + e^2\int\frac{d^4\ell}{(2\pi)^4}I_{\beta\alpha}(\ell,q,p).$$

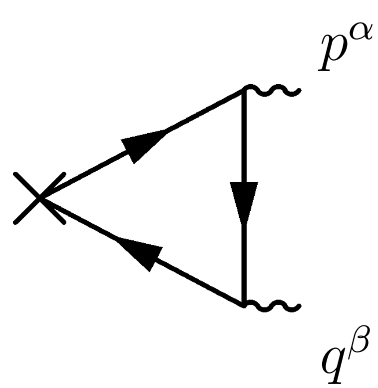
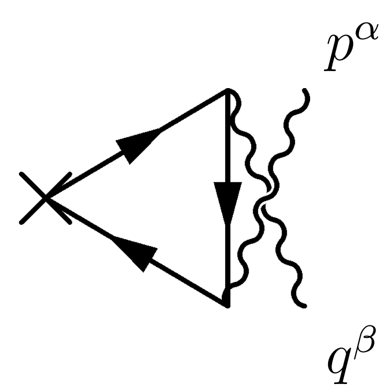
The integrand takes the form

$$
\begin{aligned}
I_{\alpha\beta}(\ell,p,q) &= i\mathrm{Tr}\left(\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right)\\
&- i\mathrm{Tr}\left(\gamma_5\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\frac{i}{\not{\ell}-m+i\epsilon}\gamma_\beta\right)\\
&+ 2m\mathrm{Tr}\left(\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right)
\end{aligned}
$$

and integrate the result over the loop momentum

$$
(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) = e^2\int\frac{d^4\ell}{(2\pi)^4}I_{\alpha\beta}(\ell,p,q) + e^2\int\frac{d^4\ell}{(2\pi)^4}I_{\beta\alpha}(\ell,q,p).
$$

The last term is the one-loop contribution to $2im\langle\overline{\psi}\gamma_5\gamma\rangle_{\mathscr{A}}$

$i\Gamma_{\mu\nu}(p,q) \equiv$  $+$ 

$\times \equiv 2m\gamma_5$

$$
\begin{aligned}
(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) &= ie^2 \int \frac{d^4\ell}{(2\pi)^4} \mathrm{Tr}\,\left(\gamma_5 \frac{i}{\not\ell - \not p - \not q - m + i\epsilon}\gamma_\beta \frac{i}{\not\ell - \not p - m + i\epsilon}\gamma_\alpha\right. \\
&\left. - \; \gamma_5 \frac{i}{\not\ell - \not q - m + i\epsilon}\gamma_\beta \frac{i}{\not\ell - m + i\epsilon}\gamma_\alpha\right) \\
&- \; ie^2 \int \frac{d^4\ell}{(2\pi)^4} \mathrm{Tr}\,\left(\gamma_5 \frac{i}{\not\ell - \not p - m + i\epsilon}\gamma_\alpha \frac{i}{\not\ell - m + i\epsilon}\gamma_\beta\right. \\
&\left. - \; \gamma_5 \frac{i}{\not\ell - \not p - \not q - m + i\epsilon}\gamma_\alpha \frac{i}{\not\ell - \not q - m + i\epsilon}\gamma_\beta\right) + 2mi\Gamma_{\alpha\beta}(p,q)
\end{aligned}
$$

and reducing the propagators:

$$
\begin{aligned}
(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) &= 4e^2 \int \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{\epsilon_{\alpha\beta\sigma\nu}(\ell - p - q)^{\sigma}(\ell - p)^{\nu}}{[(\ell - p - q)^2 - m^2 + i\epsilon][(\ell - p)^2 - m^2 + i\epsilon]} \right. \\
&\left. - \; \frac{\epsilon_{\alpha\beta\sigma\nu}(\ell - q)^{\sigma}\ell^{\nu}}{[(\ell - q)^2 - m^2 + i\epsilon](\ell^2 - m^2 + i\epsilon)} \right\} \\
&+ \; 4e^2 \int \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{\epsilon_{\alpha\beta\sigma\nu}(\ell - p)^{\sigma}\ell^{\nu}}{[(\ell - p)^2 - m^2 + i\epsilon](\ell^2 - m^2 + i\epsilon)} \right. \\
&\left. - \; \frac{\epsilon_{\alpha\beta\sigma\nu}(\ell - p - q)^{\sigma}(\ell - q)^{\nu}}{[(\ell - p - q)^2 - m^2 + i\epsilon][(\ell - q)^2 - m^2 + i\epsilon]} \right\} + 2mi\Gamma_{\alpha\beta}(p,q).
\end{aligned}
$$

$$
\begin{aligned}
(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) \;=\;& 4e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-p-q)^\sigma(\ell-p)^\nu}{[(\ell-p-q)^2-m^2+i\epsilon][(\ell-p)^2-m^2+i\epsilon]}\right. \\
&\left. -\;\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-q)^\sigma\ell^\nu}{[(\ell-q)^2-m^2+i\epsilon](\ell^2-m^2+i\epsilon)}\right\} \\
+\;& 4e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-p)^\sigma\ell^\nu}{[(\ell-p)^2-m^2+i\epsilon](\ell^2-m^2+i\epsilon)}\right. \\
-\;& \left.\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-p-q)^\sigma(\ell-q)^\nu}{[(\ell-p-q)^2-m^2+i\epsilon][(\ell-q)^2-m^2+i\epsilon]}\right\}+2mi\Gamma_{\alpha\beta}(p,q).
\end{aligned}
$$

The two integrals are **linearly divergent** and have the structure

$$
\int\frac{d^4\ell}{(2\pi)^4}\Big[f^\mu(\ell+\xi)-f^\mu(\ell)\Big]=\frac{iC}{32\pi^2}\xi^\mu.
$$

remember

$$
f^\mu(\ell_E)\sim C\frac{\ell_E^\mu}{\ell_E^4}
$$

with

$$
\xi^\mu=-p^\mu
$$

$$
\xi^\mu=q^\mu
$$

respectively.

We find the result

$$(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) \;=\; \frac{ie^2}{4\pi^2}\epsilon_{\alpha\beta\sigma\nu}p^\sigma q^\nu + 2mi\Gamma_{\alpha\beta}(p,q).$$

The axial Ward identity is violated in the limit $m \to 0$.

The axial-vector symmetry is anomalous!

But not so fast… what happens with the **vector current**?

$$\begin{aligned} p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) \;&=\; e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell - m + i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\not p\right) \\ &+\; e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell - m + i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\not p\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\right)\end{aligned}$$

$$
\begin{aligned}
p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\,\left(\frac{i}{\not\ell-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\not p\right)\\
&+ e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\,\left(\frac{i}{\not\ell-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\not p\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\right)
\end{aligned}
$$

Using the identities

$$
\not p = (\not\ell-m)-(\not\ell-\not p-m), \qquad \not p = -(\not\ell-\not p-\not q-m)+(\not\ell-\not q-m)
$$

we have

$$
\begin{aligned}
p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\,\left(\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\right.\\
&\qquad\left. - \gamma_\mu\gamma_5\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right)\\
&- ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\,\left(\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right.\\
&\qquad\left. - \gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right).
\end{aligned}
$$

$$
\begin{aligned}
p^{\alpha} i\Gamma_{\mu\alpha\beta}(p,q) &= e^2 \int \frac{d^4\ell}{(2\pi)^4} \mathrm{Tr}\left(\frac{i}{\not{\ell} - m + i\epsilon}\gamma_\mu\gamma_5 \frac{i}{\not{\ell} - \not{p} - \not{q} - m + i\epsilon}\gamma_\beta \frac{i}{\not{\ell} - \not{p} - m + i\epsilon}\not{p}\right) \\
&+ e^2 \int \frac{d^4\ell}{(2\pi)^4} \mathrm{Tr}\left(\frac{i}{\not{\ell} - m + i\epsilon}\gamma_\mu\gamma_5 \frac{i}{\not{\ell} - \not{p} - \not{q} - m + i\epsilon}\not{p} \frac{i}{\not{\ell} - \not{q} - m + i\epsilon}\gamma_\beta\right)
\end{aligned}
$$

Using the identities

$$
\not{p} = (\not{\ell} - m) - (\not{\ell} - \not{p} - m), \qquad \not{p} = -(\not{\ell} - \not{p} - \not{q} - m) + (\not{\ell} - \not{q} - m)
$$

we have

$$
\begin{aligned}
p^{\alpha} i\Gamma_{\mu\alpha\beta}(p,q) &= ie^2 \int \frac{d^4\ell}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\mu\gamma_5 \frac{i}{\not{\ell} - \not{p} - \not{q} - m + i\epsilon}\gamma_\beta \frac{i}{\not{\ell} - \not{p} - m + i\epsilon}\right. \\
&- \left.\gamma_\mu\gamma_5 \frac{i}{\not{\ell} - \not{q} - m + i\epsilon}\gamma_\beta \frac{i}{\not{\ell} - m + i\epsilon}\right) \\
&- ie^2 \int \frac{d^4\ell}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\mu\gamma_5 \frac{i}{\not{\ell} - \not{p} - \not{q} - m + i\epsilon}\gamma_\beta \frac{i}{\not{\ell} - m + i\epsilon}\right. \\
&- \left.\gamma_\mu\gamma_5 \frac{i}{\not{\ell} - \not{p} - \not{q} - m + i\epsilon}\gamma_\beta \frac{i}{\not{\ell} - m + i\epsilon}\right).
\end{aligned}
$$

$= 0$

(no shift required)

The remaining integral

$$\begin{aligned}
p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\right.\\
&\quad\left.-\ \gamma_\mu\gamma_5\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right)\\
&= 4e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{\epsilon_{\mu\beta\sigma\nu}(\ell-p-q)^\sigma(\ell-p)^\nu}{[(\ell-p-q)^2-m^2+i\epsilon][(\ell-p)^2-m^2+i\epsilon]}\right.\\
&\quad\left.-\ \frac{\epsilon_{\mu\beta\sigma\nu}(\ell-q)^\sigma\ell^\nu}{[(\ell-q)^2-m^2+i\epsilon](\ell^2-m^2+i\epsilon)}\right\}
\end{aligned}$$

has again the structure

$$\int\frac{d^4\ell}{(2\pi)^4}\Big[f^\mu(\ell+\xi)-f^\mu(\ell)\Big]=\frac{iC}{32\pi^2}\xi^\mu.$$

with

$$\xi^\mu=-p^\mu$$

The computation shows that **the gauge Ward identity is violated**!

$$p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) = -\frac{ie^2}{8\pi^2}\epsilon_{\mu\beta\sigma\nu}p^\sigma q^\nu$$

The computation shows that **the gauge Ward identity is violated**!

$$p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) = -\frac{ie^2}{8\pi^2}\epsilon_{\mu\beta\sigma\nu}p^\sigma q^\nu$$

But remember the **ambiguity** in parametrizing the loop momentum. It seems we made the **wrong choice**…

Changing the parametrization

$$\ell^\mu \longrightarrow \ell^\mu + \alpha p^\mu + \beta q^\mu$$

introduces a **change** in the amplitude

$$i\Gamma_{\mu\alpha\beta}(p,q) \longrightarrow i\Gamma_{\mu\alpha\beta}(p,q) + \Delta_{\mu\alpha\beta}(\alpha,\beta)$$

Can we select α and β so the vector Ward identity is enforced?

Luckily, we don't have to redo the whole computation! Imposing:

- **Parity**
- **Lorentz invariance**
- **Bose symmetry**

and remembering that the ambiguity **does not depend on masses**, we only have one possibility for the change in the amplitude

$$i\Gamma_{\mu\alpha\beta}(p,q) \longrightarrow i\Gamma_{\mu\alpha\beta}(p,q) + \frac{ie^2}{8\pi^2} a\epsilon_{\mu\alpha\beta\sigma}(p-q)^\sigma$$

where $a = a(\alpha,\beta)$

Using now our results for the triangle diagrams

$$
\begin{aligned}
(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) &= \frac{ie^2}{4\pi^2}(1-a)\epsilon_{\alpha\beta\sigma\nu}p^{\sigma}q^{\nu} + 2mi\Gamma_{\alpha\beta}(p,q), \\
p^{\alpha} i\Gamma_{\mu\alpha\beta}(p,q) &= -\frac{ie^2}{8\pi^2}(1+a)\epsilon_{\alpha\beta\sigma\nu}p^{\sigma}q^{\nu}.
\end{aligned}
$$

Thus, the **physically correct choice** is to take $a=-1$ for which

$$
\begin{aligned}
p^{\alpha} i\Gamma_{\mu\alpha\beta}(p,q) &= 0, \\
(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) &= \frac{ie^2}{2\pi^2}\epsilon_{\alpha\beta\sigma\nu}p^{\sigma}q^{\nu} + 2mi\Gamma_{\alpha\beta}(p,q)
\end{aligned}
$$

The axial-vector current is anomalous!

It is important that there is **no value** of a for which **both** Ward identities are satisfied **simultaneously**.

In our calculation we **did not commit** to any particular **regularization method** (in fact, we didn't have to), only to the preservation of gauge invariance.

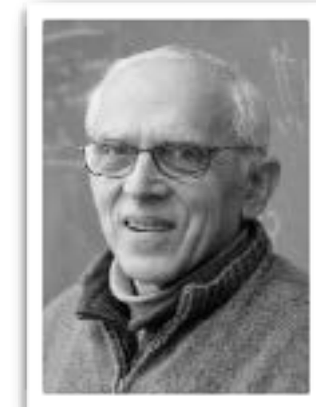
The result for the axial anomaly can be obtained computing the triangle diagram using any regularization method that **preserves gauge invariance**: e.g.

- Pauli-Villars (see Bertlmann)
- Dimensional regularization, but beware of γ_5 (see Peskin & Schroeder)
- Point-splitting (wait and see)
- Dispersion relations (see Bertlmann)

$$\vdots$$

Transforming the result back to position space,

$$\begin{aligned}\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} &= \frac{i}{2}\int d^4y_1 d^4y_2 \mathscr{A}^\alpha(y_1)\mathscr{A}^\beta(y_2) \\ &\times \int \frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q)e^{ip\cdot(y_1-x)+iq\cdot(y_2-x)}.\end{aligned}$$



Steven Adler
(b. 1939)

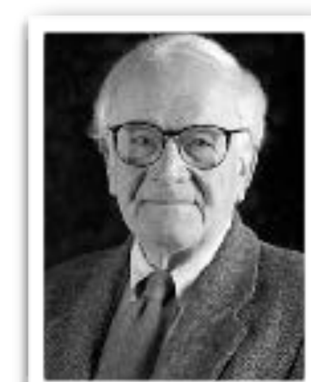
we arrive at the celebrated **Adler-Bell-Jackiw anomaly**

Jack Steinberger
(b. 1921)

$$\boxed{\partial_\mu \langle J_{A}^\mu(x)\rangle_{\mathscr{A}} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta} + 2im\langle\overline{\psi}(x)\gamma_5\psi(x)\rangle_{\mathscr{A}}}$$

John S. Bell
(1928-1990)

Julian Schwinger
(1918-1994)



Roman Jackiw
(b. 1939)