Shedding light on our calculation

Why didn't we have to commit to any particular regularization?

The amplitude $i\Gamma_{\mu\alpha\beta}(p,q)$ should satisfy a number of condition:

- **Parity**: begin parity odd, it should contain an $\epsilon_{\mu\nu\alpha\beta}$ tensor
- **Poincaré invariance**: it should be a rank-three tensor depending only on p and q

This forces the following **general structure** for the amplitude

$$\begin{aligned} i\Gamma_{\mu\alpha\beta}(p,q) &= f_1\epsilon_{\mu\alpha\beta\sigma}p^\sigma + f_2\epsilon_{\mu\alpha\beta\sigma}q^\sigma + f_3\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda \\ &+ f_4\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda + f_5\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda \\ &+ f_6\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda + f_7\epsilon_{\alpha\beta\sigma\lambda}p_\mu p^\sigma q^\lambda + f_8\epsilon_{\alpha\beta\sigma\lambda}q_\mu p^\sigma q^\lambda \end{aligned}$$

$$f_i \equiv f_i(p,q)$$

where f_i are scalar functions of the momenta. Moreover, using

$$\epsilon_{\alpha\beta\sigma\lambda}w_\mu + \epsilon_{\beta\sigma\lambda\mu}w_\alpha + \epsilon_{\sigma\lambda\mu\alpha}w_\beta + \epsilon_{\lambda\mu\alpha\beta}w_\sigma + \epsilon_{\mu\alpha\beta\sigma}w_\lambda = 0,$$

we can absorb f_7 and f_8 into the other f's.

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q) &= f_1\epsilon_{\mu\alpha\beta\sigma}p^\sigma + f_2\epsilon_{\mu\alpha\beta\sigma}q^\sigma + f_3\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda \\
&+ f_4\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda + f_5\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda + f_6\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda
\end{aligned}
$$

- **Bose symmetry**: it should satisfy $i\Gamma_{\mu\alpha\beta}(p,q) = i\Gamma_{\mu\beta\alpha}(q,p)$

This imposes the following conditions on the coefficients

$$
f_1(p,q) = -f_2(q,p), \qquad f_3(p,q) = -f_6(q,p), \qquad f_4(p,q) = -f_5(q,p).
$$

Let's do a bit of **dimensional analysis**:

$$
[i\Gamma_{\mu\alpha\beta}] = E \quad \longrightarrow \quad \begin{cases} [f_1] = [f_2] = E^0 \\ [f_3] = \ldots = [f_6] = E^{-2} \end{cases}
$$

By power counting, f_1 and f_2 are **logarithmically divergent** integrals while f_3 ,…, f_6 are **convergent integrals.**

$$\begin{aligned} i\Gamma_{\mu\alpha\beta}(p,q) &= f_1\epsilon_{\mu\alpha\beta\sigma}p^\sigma + f_2\epsilon_{\mu\alpha\beta\sigma}q^\sigma + f_3\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda \\ &+ f_4\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda + f_5\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda + f_6\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda \end{aligned}$$

All **ambiguities** in the amplitude are confined to the coefficients f_1 and f_2.

Next we look at the contractions

$$\begin{aligned} p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= \Big(f_2 - p^2 f_5 - p\cdot q f_6\Big)\epsilon_{\mu\beta\alpha\sigma}q^\alpha p^\sigma, \\ q^\beta i\Gamma_{\mu\alpha\beta}(p,q) &= \Big(f_1 - q^2 f_4 - p\cdot q f_3\Big)\epsilon_{\mu\alpha\beta\sigma}q^\beta p^\sigma, \\ (p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) &= \Big(-f_1 + f_2\Big)\epsilon_{\alpha\beta\sigma\lambda}q^\sigma p^\lambda. \end{aligned}$$

Imposing the vector (gauge) Ward identities

$$p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) = 0 = q^\beta i\Gamma_{\mu\alpha\beta}(p,q)$$

completely fixes the ambiguous integrals in terms of finite ones

$$f_1(p,q) = q^2 f_4(p,q) - p\cdot q f_3(p,q)$$

$$f_2(p,q) = p^2 f_5(p,q) - p\cdot q f_6(p,q)$$

Using these identities, the axial anomaly is given by

$$(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) = \Big[p^2 f_5 - q^2 f_4 + p\cdot q(-f_3+f_6)\Big]\epsilon_{\alpha\beta\sigma\lambda}q^{\sigma}p^{\lambda}$$

With these general considerations we **learn** a number of things:

- All **ambiguities** in the triangle diagram are codified in **nominally logarithmically divergent integrals**.
- These are **completely fixed** by requiring the **conservation of the gauge current**.
- Once this is done, the **axial anomaly** is given by **finite** integrals.

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- These are **comp** … **of the gauge curr** …

- Once this is done, …

why **logarithmically** divergent?

$$\int \frac{d^4\ell}{(2\pi)^4}\Big[f^\mu(\ell+\xi)-f^\mu(\ell)\Big] = \xi^\alpha \int \frac{d^4\ell}{(2\pi)^4}\,\frac{\partial f^\mu}{\partial \ell^\alpha}\bigg|_{\xi=0}$$

$f^\mu(\ell+\xi)$, $f^\mu(\ell)$ $\sim \mathcal{O}\left(\frac{1}{\ell^3}\right)$; $\frac{\partial f^\mu}{\partial \ell^\alpha}$ $\sim \mathcal{O}\left(\frac{1}{\ell^4}\right)$

Using these identities, the axial anomaly is given by

$$(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) = \Big[p^2 f_5 - q^2 f_4 + p\cdot q(-f_3+f_6)\Big]\epsilon_{\alpha\beta\sigma\lambda}q^{\sigma}p^{\lambda}$$

With these general considerations we **learn** a number of things:

- All **ambiguities** in the triangle diagram are codified in **nominally logarithmically divergent integrals**.
- These are **completely fixed** by requiring the **conservation of the gauge current**.
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An alternative procedure

The anomaly can be reobtained using a point-splitting regularization of the axial-vector current **composite operator**

$$J^{\mu}_{\mathrm{A}}(x)_{\mathrm{reg}} = \overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma_{\mu}\gamma_{5}\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_{\alpha}\mathscr{A}^{\alpha}(y)\right]$$

where $a \in \mathbb{R}$ and ϵ^{μ} satisfies $\epsilon^{0} > 0$

Under a gauge transformation

$$\left.\begin{aligned}
\psi\left(x-\frac{\epsilon}{2}\right) &\longrightarrow e^{i\alpha\left(x-\frac{\epsilon}{2}\right)}\psi\left(x-\frac{\epsilon}{2}\right) \\
\overline{\psi}\left(x+\frac{\epsilon}{2}\right) &\longrightarrow e^{-i\alpha\left(x+\frac{\epsilon}{2}\right)}\overline{\psi}\left(x+\frac{\epsilon}{2}\right) \\
\mathscr{A}_{\mu}(y) &\longrightarrow \mathscr{A}_{\mu}(y)+\frac{1}{e}\partial_{\mu}\epsilon(y)
\end{aligned}\right\} \quad J^{\mu}_{\mathrm{A}}(x)_{\mathrm{reg}} \longrightarrow e^{i(a-1)\left[\alpha\left(x+\frac{\epsilon}{2}\right)-\alpha\left(x-\frac{\epsilon}{2}\right)\right]}J^{\mu}_{\mathrm{A}}(x)_{\mathrm{reg}}$$

The regularization is gauge invariant only for $a = 1$

$$J^\mu_{\rm A}(x)_{\rm reg} = \overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_\alpha \mathscr{A}^\alpha(y)\right]$$

We compute now its divergence

$$\begin{aligned}\partial_\mu J^\mu_{\rm A}(x)_{\rm reg} = {}& \partial_\mu\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_\alpha \mathscr{A}^\alpha(y)\right] \\ & + \overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\partial_\mu\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_\alpha \mathscr{A}^\alpha(y)\right] \\ & + ie\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)\left[a\partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2} dy_\alpha \mathscr{A}^\alpha(y)\right] \\ & \times\ \exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_\alpha \mathscr{A}^\alpha(y)\right]\end{aligned}$$

and use the fermion EOM

$$i\gamma^\mu\partial_\mu\psi = m\psi - e\mathscr{A}_\mu\gamma^\mu\psi \qquad\qquad -i\partial_\mu\overline{\psi}\gamma^\mu = m\overline{\psi} - e\overline{\psi}\gamma^\mu\mathscr{A}_\mu$$

$$\partial_\mu J^\mu_{\rm A}(x)_{\rm reg} = 2im[\overline{\psi}\gamma_5\psi]_{\rm reg} - ie\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)$$

$$\times\ \left[\mathscr{A}_\mu\left(x+\frac{\epsilon}{2}\right) - \mathscr{A}_\mu\left(x-\frac{\epsilon}{2}\right) - a\partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2} dy^\alpha \mathscr{A}_\alpha(y)\right]$$

$$\times\ \exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_\alpha \mathscr{A}^\alpha(y)\right]$$

Identifying $J^\mu_{\rm A}(x)_{\rm reg}$ and expanding to first order in ϵ^μ we have

$$\partial_\mu J^\mu_{\rm A}(x)_{\rm reg} = 2im[\overline{\psi}\gamma_5\psi]_{\rm reg} - iJ^\mu_{\rm A}(x)_{\rm reg}\epsilon^\alpha\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

and now compute its vacuum expectation value

$$\partial_\mu\langle J^\mu_{\rm A}(x)_{\rm reg}\rangle_{\mathscr{A}} = 2im\langle[\overline{\psi}\gamma_5\psi]_{\rm reg}\rangle_{\mathscr{A}} - ie\epsilon^\alpha\langle J^\mu_{\rm A}(x)_{\rm reg}\rangle_{\mathscr{A}}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

$$\partial_\mu J^\mu_\mathrm{A}(x)_\mathrm{reg} = 2im[\overline{\psi}\gamma_5\psi]_\mathrm{reg} - ie\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)$$

$$\times\left[\mathscr{A}_\mu\left(x+\frac{\epsilon}{2}\right) - \mathscr{A}_\mu\left(x-\frac{\epsilon}{2}\right) - a\partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2} dy^\alpha\mathscr{A}_\alpha(y)\right]$$

$$\times\ \exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_\alpha\mathscr{A}^\alpha(y)\right]$$

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$$\partial_\mu\langle J^\mu_\mathrm{A}(x)_\mathrm{reg}\rangle_\mathscr{A} = 2im\langle[\overline{\psi}\gamma_5\psi]_\mathrm{reg}\rangle_\mathscr{A} - ie\epsilon^\alpha\langle J^\mu_\mathrm{A}(x)_\mathrm{reg}\rangle_\mathscr{A}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

Next, we evaluate $\langle J^{\mu}_{\rm A}(x)_{\rm reg}\rangle_{\mathscr{A}}$

$$\langle J^{\mu}_{\rm A}(x)_{\rm reg}\rangle_{\mathscr{A}} = \left\langle \overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^{\mu}\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)\right\rangle_{\mathscr{A}} \exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_{\alpha}\mathscr{A}^{\alpha}(y)\right]$$

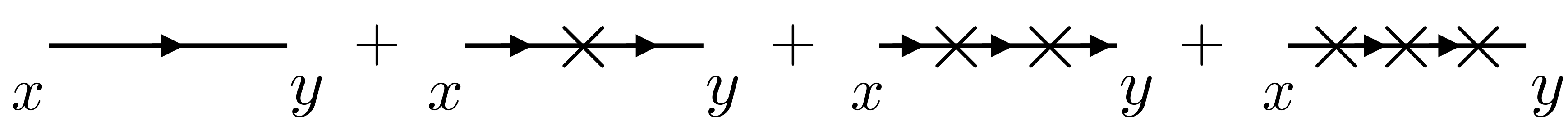
$\Downarrow$ $\epsilon^0 > 0$

$$\gamma^{\mu}_{ab}\gamma_{5\,bc}\left\langle T\left[\overline{\psi}_a\left(x+\frac{\epsilon}{2}\right)\psi_c\left(x-\frac{\epsilon}{2}\right)\right]\right\rangle_{\mathscr{A}}$$

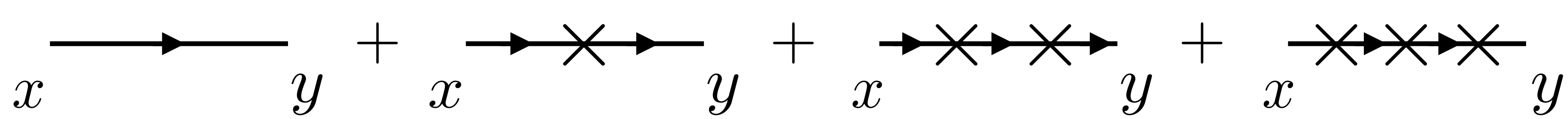
$$= -\mathrm{Tr}\left\{\gamma^{\mu}\gamma_5\left\langle T\left[\psi\left(x-\frac{\epsilon}{2}\right)\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\right]\right\rangle_{\mathscr{A}}\right\}$$

$$= -\mathrm{Tr}\left[\gamma^{\mu}\gamma_5 G\left(x-\frac{\epsilon}{2}, x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]$$

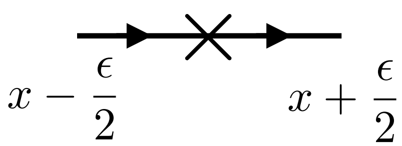
where the propagator can be computed diagrammatically as:

$G(x,y)_{\mathscr{A}} =$ 

$+ \ldots$

$G(x,y)_{\mathscr{A}} =$  $+ \ldots$

We look at the term **linear in the gauge field**:



$$= ie\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\left(\frac{i}{\not{p}+\frac{1}{2}\not{q}-m}\gamma^{\mu}\frac{i}{\not{p}-\frac{1}{2}\not{q}-m}\right)e^{-iq\cdot x}e^{ip\cdot\epsilon}\mathscr{A}_{\mu}(q)$$

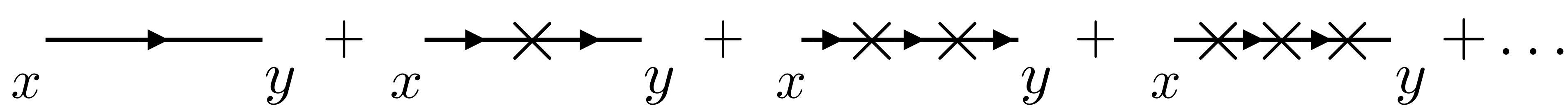
With this we go back to

$$\partial_{\mu}\langle J^{\mu}_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}} = 2im\langle[\overline{\psi}\gamma_5\psi]_{\mathrm{reg}}\rangle_{\mathscr{A}} - ie\epsilon^{\alpha}\langle J^{\mu}_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}}\Big(\partial_{\alpha}\mathscr{A}_{\mu} - a\partial_{\mu}\mathscr{A}_{\alpha} + \ldots\Big)$$

and

$$\langle J^{\mu}_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = -\mathrm{Tr}\left[\gamma^{\mu}\gamma_5 G\left(x-\frac{\epsilon}{2}, x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_{\alpha}\mathscr{A}^{\alpha}(y)\right]$$

$$G(x,y)_{\mathscr{A}} =$$



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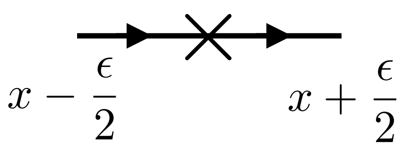

$$= ie\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\left(\frac{i}{\not{p}+\frac{1}{2}\not{q}-m}\gamma^\mu\frac{i}{\not{p}-\frac{1}{2}\not{q}-m}\right)e^{-iq\cdot x}e^{ip\cdot\epsilon}\mathscr{A}_\mu(q)$$

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$$\partial_\mu\langle J^\mu_{\rm A}(x)_{\rm reg}\rangle_{\mathscr{A}} = 2im\langle[\overline{\psi}\gamma_5\psi]_{\rm reg}\rangle_{\mathscr{A}} - ie\epsilon^\alpha\langle J^\mu_{\rm A}(x)_{\rm reg}\rangle_{\mathscr{A}}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha+\ldots\Big)$$

and

$$\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -{\rm Tr}\left[\gamma^\mu\gamma_5 G\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\alpha\mathscr{A}^\alpha(y)\right]$$

Furthermore, we use

$$\epsilon^\alpha e^{ip\cdot\epsilon} = -i\frac{\partial}{\partial p_\alpha}e^{ip\cdot\epsilon}$$ ⟹ integration by parts

$$-\mathrm{Tr}\left[\epsilon^\alpha\gamma^\mu\gamma_5 G\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]=$$

$$=e\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\mathscr{A}_\nu(q)\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot\epsilon}\frac{\partial}{\partial p_\alpha}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{i}{\not p+\frac{1}{2}\not q-m}\gamma^\nu\frac{i}{\not p-\frac{1}{2}\not q-m}\right)$$

$$\epsilon^\mu \longrightarrow 0$$

$$\lim_{\epsilon\to 0}\epsilon^\alpha\langle J^\mu_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}}=-\lim_{\epsilon\to 0}\mathrm{Tr}\left[\epsilon^\alpha\gamma^\mu\gamma_5 G\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]=$$

$$=\frac{ie}{16\pi^2}\epsilon^{\mu\alpha\nu\sigma}\mathscr{F}_{\nu\sigma}(x)$$

With this result we return to the regularized anomaly

$$\partial_\mu\langle J^\mu_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}}=2im\langle[\overline{\psi}\gamma_5\psi]_{\mathrm{reg}}\rangle_{\mathscr{A}}-ie\epsilon^\alpha\langle J^\mu_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}}\Big(\partial_\alpha\mathscr{A}_\mu-a\partial_\mu\mathscr{A}_\alpha+\ldots\Big)$$

$$-\mathrm{Tr}\left[\epsilon^\alpha\gamma^\mu\gamma_5 G\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]=$$

$$=e\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\mathscr{A}_\nu(q)\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot\epsilon}\frac{\partial}{\partial p_\alpha}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{i}{\not{p}+\frac{1}{2}\not{q}-m}\gamma^\nu\frac{i}{\not{p}-\frac{1}{2}\not{q}-m}\right)$$

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$$\partial_\mu \langle J_A^\mu(x)_{\rm reg}\rangle_{\mathscr{A}} = 2im\langle[\overline{\psi}\gamma_5\psi]_{\rm reg}\rangle_{\mathscr{A}} - ie\epsilon^\alpha\langle J_A^\mu(x)_{\rm reg}\rangle_{\mathscr{A}}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

$$\lim_{\epsilon\to 0}\epsilon^\alpha\langle J_A^\mu(x)_{\rm reg}\rangle_{\mathscr{A}} = \frac{ie}{16\pi^2}\epsilon^{\mu\alpha\nu\sigma}\mathscr{F}_{\nu\sigma}(x)$$

Using the simple identity

$$\epsilon^{\mu\alpha\nu\sigma}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha\Big) = (1+a)\epsilon^{\mu\alpha\nu\sigma}\partial_\alpha\mathscr{A}_\mu = \frac{1+a}{2}\epsilon^{\mu\alpha\nu\sigma}\mathscr{F}_{\alpha\mu}$$

we arrive at the result

$$\lim_{\epsilon\to 0}\partial_\mu\langle J_A^\mu(x)_{\rm reg}\rangle_{\mathscr{A}} = 2im\lim_{\epsilon\to 0}\langle[\overline{\psi}\gamma_5\psi]_{\rm reg}\rangle_{\mathscr{A}} + \frac{e^2}{32\pi^2}(1+a)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

$$\lim_{\epsilon\to 0}\partial_\mu\langle J^\mu_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}} = 2im\lim_{\epsilon\to 0}\langle[\overline{\psi}\gamma_5\psi]_{\mathrm{reg}}\rangle_{\mathscr{A}} + \frac{e^2}{32\pi^2}(1+a)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

We can repeat the same calculation for the **vector current**

$$J^\mu_V(x)_{\mathrm{reg}} = \overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\alpha\mathscr{A}^\alpha(y)\right]$$

whose divergence is given by

$$\lim_{\epsilon\to 0}\partial_\mu\langle J^\mu_V(x)_{\mathrm{reg}}\rangle_{\mathscr{A}} = \frac{e^2}{64\pi^2}(1-a)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

Thus, we have arrived at the result:

For $a=1$

We recover the ABJ **anomaly** and the vector current is conserved

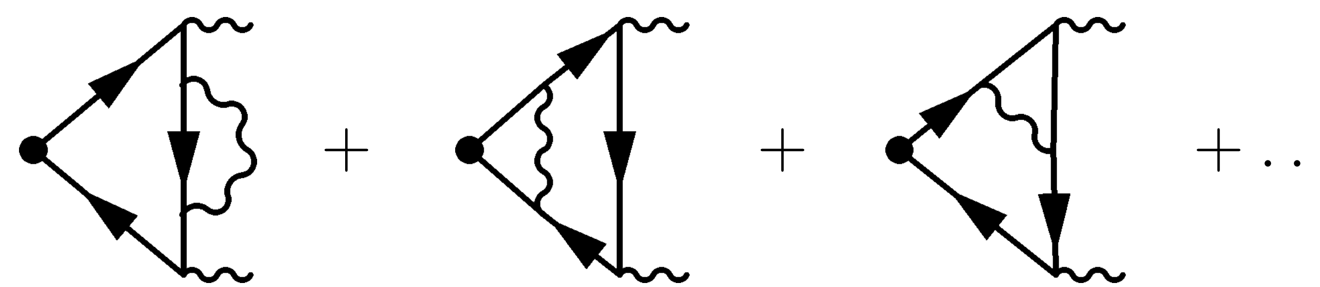
For $a=-1$

The axial-vector current is conserved but gauge invariance is broken.

Quantum corrections

What about higher loops?

The ABJ anomaly is a one-loop result. Is it corrected by higher loop diagrams? E.g.



These diagrams contain **five** fermion propagator. The integration over the fermion loop momentum

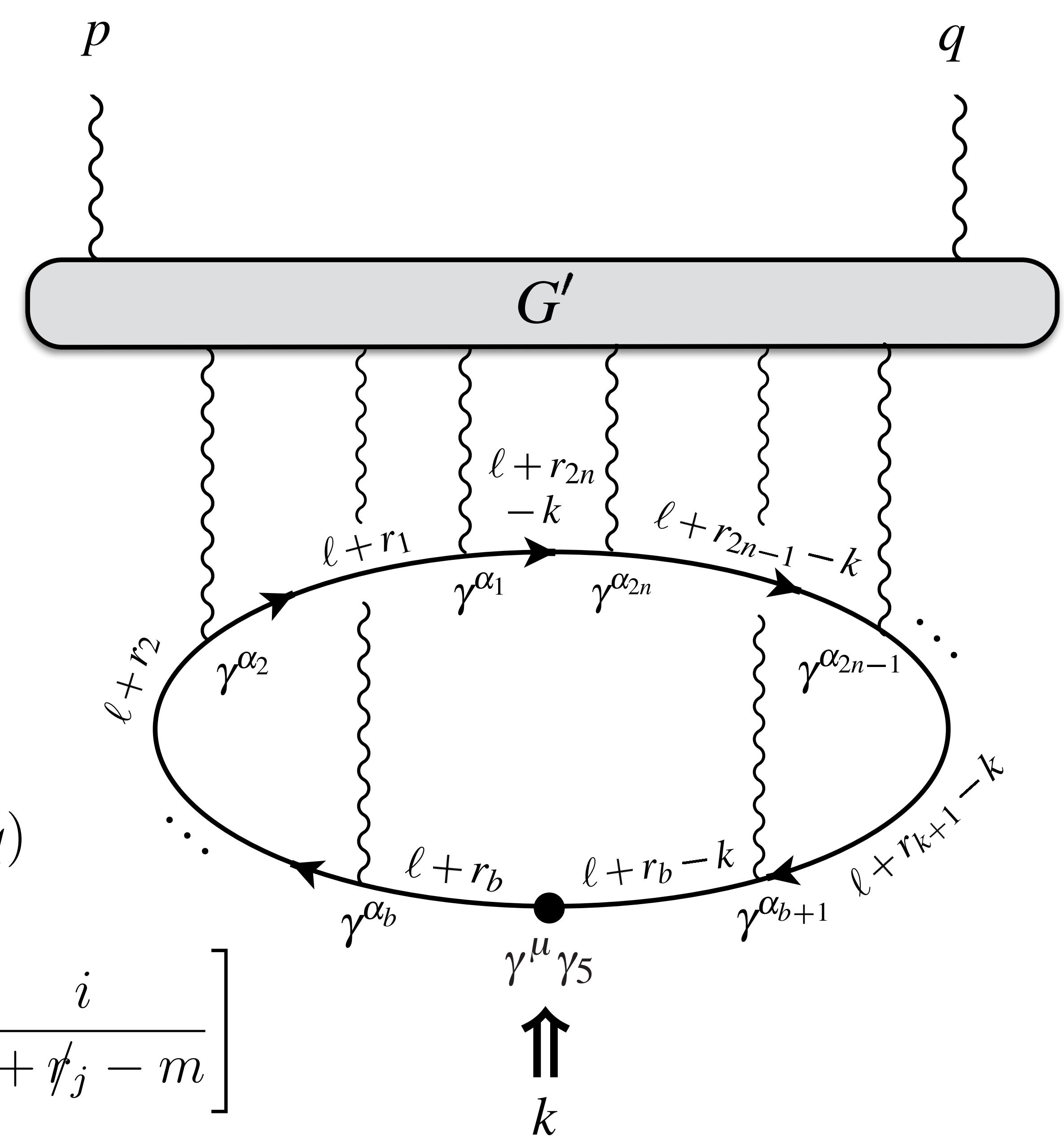
$$\cdots \int \frac{d^4\ell}{(2\pi)^4} \prod_{i=1}^{5} \frac{i}{\not{\ell} + \not{\Delta}_i + i\varepsilon} \cdots$$

is convergent. The remaining loops can be handled using a gauge invariant regulator, for example

$$\Delta S = \frac{1}{\Lambda^2} \int d^4x F_{\mu\nu} \Box F^{\mu\nu} \quad \Longrightarrow \quad G_{\mu\nu}(p) \sim \frac{\Lambda^2}{p^4}$$

This heuristic argument can be made more precise.

Consider a generic topology contributing to the divergence of the axial-vector current:



$$
\begin{aligned}
k^\mu i\Gamma_{\mu\alpha\beta}(p,q)_{L\text{-loop}} \;&=\; \int \prod_{a=1}^{L-1}\frac{d^4\ell_a}{(2\pi)^4}\Gamma^{(G')}_{\alpha\beta}(r_1,\ldots,r_{2n};p,q) \\
&\times\; \int\frac{d^4\ell}{(2\pi)^4}\sum_{b=1}^{2n}\mathrm{Tr}\,\left\{\left[\prod_{j=1}^{b-1}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}\right]\right. \\
&\times\;(-ie\gamma^{\alpha_b})\frac{i}{\not\ell+\not r_b-m}ik_\mu\gamma^\mu\gamma_5\frac{i}{\not\ell+\not r_b-\not k-m} \\
&\times\;\left.\left[\prod_{j=b+1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]\right\}.
\end{aligned}
$$

$$k^\mu i\Gamma_{\mu\alpha\beta}(p,q)_{L\text{-loop}} = \int \prod_{a=1}^{L-1} \frac{d^4\ell_a}{(2\pi)^4}\Gamma^{(G')}_{\alpha\beta}(r_1,\ldots,r_{2n};p,q)\int \frac{d^4\ell}{(2\pi)^4}\sum_{b=1}^{2n}\mathrm{Tr}\,\left\{\left[\prod_{j=1}^{b-1}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-m}\right]\right.$$

$$\left.\times\,(-ie\gamma^{\alpha_b})\frac{i}{\not{\ell}+\not{r}_b-m}ik_\mu\gamma^\mu\gamma_5\frac{i}{\not{\ell}+\not{r}_b-k-m}\left[\prod_{j=b+1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]\right\}.$$

We simplify this expression using,

$$\not{k}\gamma_5 = (\not{\ell}+\not{r}_b-m)\gamma_5+\gamma_5(\not{\ell}+\not{r}_b-\not{k}-m)+2m\gamma_5$$

to write

$$\begin{aligned}\frac{i}{\not{\ell}+\not{r}_b-m}ik_\mu\gamma^\mu\gamma_5\frac{i}{\not{\ell}+\not{r}_b-k-m} &= \frac{i}{\not{\ell}+\not{r}_b-m}(2im\gamma_5)\frac{i}{\not{\ell}+\not{r}_b-k-m}\\ &- \frac{i}{\not{\ell}+\not{r}_b-m}\gamma_5-\gamma_5\frac{i}{\not{\ell}+\not{r}_b-\not{k}-m}.\end{aligned}$$

$$k^\mu i\Gamma_{\mu\alpha\beta}(p,q)_{L\text{-loop}} = \int\prod_{a=1}^{L-1}\frac{d^4\ell_a}{(2\pi)^4}\Gamma^{(G')}_{\alpha\beta}(r_1,\ldots,r_{2n};p,q)\int\frac{d^4\ell}{(2\pi)^4}\sum_{b=1}^{2n}\mathrm{Tr}\left\{\left[\prod_{j=1}^{b-1}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}\right]\right.$$

$$\left.\times\,(-ie\gamma^{\alpha_b})\frac{i}{\not\ell+\not r_b-m}ik_\mu\gamma^\mu\gamma_5\frac{i}{\not\ell+\not r_b-k-m}\left[\prod_{j=b+1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]\right\}.$$

We simplify this expression using,

$$\not k\gamma_5 = (\not\ell+\not r_b-m)\gamma_5+\gamma_5(\not\ell+\not r_b-\not k-m)+2m\gamma_5$$

to write

$$\frac{i}{\not\ell+\not r_b-m}ik_\mu\gamma^\mu\gamma_5\frac{i}{\not\ell+\not r_b-k-m} = \frac{i}{\not\ell+\not r_b-m}(2im\gamma_5)\frac{i}{\not\ell+\not r_b-k-m} - \frac{i}{\not\ell+\not r_b-m}\gamma_5-\gamma_5\frac{i}{\not\ell+\not r_b-\not k-m}.$$

Thus, the result has the structure:

$$k^\mu i\Gamma_{\mu\alpha\beta}(p,q)_{L\text{-loop}} = 2mi\Gamma_{\alpha\beta}(p,q)_{L\text{-loop}} + \Delta_{\alpha\beta}(p,q).$$

The relevant term contributing to $\Delta_{\alpha\beta}(p,q)$ is

$$-\sum_{b=1}^{2n}\mathrm{tr}\left\{\left[\prod_{j=1}^{b}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}\right]i\gamma_5\left[\prod_{j=b+1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]\right.$$
$$\left.-\left[\prod_{j=1}^{b-1}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}\right]i\gamma_5\left[\prod_{j=b}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]\right\}.$$

and most terms cancel

$$-\mathrm{tr}\left\{(-ie\gamma^{\alpha_1})\frac{i}{\not\ell+\not r_1-m}i\gamma_5\left[\prod_{j=2}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]-i\gamma_5\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]\right.$$
$$+\left[\prod_{j=1}^{2}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}\right]i\gamma_5\left[\prod_{j=3}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]$$
$$\left.-(-ie\gamma^{\alpha_1})\frac{i}{\not\ell+\not r_1-m}i\gamma_5\left[\prod_{j=2}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]+\ldots\right\}$$

Thus, the result has the structure:

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The relevant term contributing to $\Delta_{\alpha\beta}(p,q)$ is

$$-\sum_{b=1}^{2n} \mathrm{tr}\left\{\left[\prod_{j=1}^{b}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-m}\right] i\gamma_5 \left[\prod_{j=b+1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]\right.$$
$$\left. - \left[\prod_{j=1}^{b-1}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-m}\right] i\gamma_5 \left[\prod_{j=b}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]\right\}.$$

and most terms cancel

$$-\mathrm{tr}\left\{(-ie\gamma^{\alpha_1})\frac{i}{\not{\ell}+\not{r}_1-m} i\gamma_5 \left[\prod_{j=2}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right] - i\gamma_5\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]\right.$$
$$+ \left[\prod_{j=1}^{2}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-m}\right] i\gamma_5 \left[\prod_{j=3}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]$$
$$\left. - (-ie\gamma^{\alpha_1})\frac{i}{\not{\ell}+\not{r}_1-m} i\gamma_5 \left[\prod_{j=2}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right] + \ldots\right\}$$

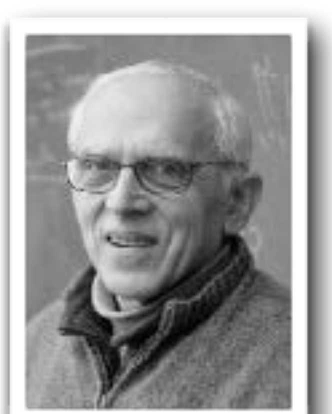
The only **surviving terms** are

$$-\mathrm{tr}\left\{\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}\right]i\gamma_5-i\gamma_5\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell-\not r_j-\not k-m}\right]\right\}$$

Hence, the final result for the anomalous piece is:

$$\begin{aligned}\Delta_{\alpha\beta}(p,q) &= -\int\prod_{a=1}^{L-1}\frac{d^4\ell_a}{(2\pi)^4}\Gamma^{(G')}_{\alpha\beta}(r_1,\ldots,r_{2n};p,q)\\ &\times \int\frac{d^4\ell}{(2\pi)^4}\sum_{b=1}^{2n}\mathrm{Tr}\left\{i\gamma_5\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}-\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]\right\}\end{aligned}$$

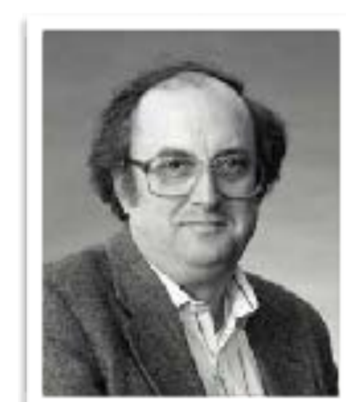
For $n>1$ we can shift the integration momentum and cancel the terms.



Steven Adler
(b. 1939)

The ABJ anomaly does not receive quantum corrections

(**Adler-Bardeen theorem**)



William A. Bardeen
(b. 1941)

UV or IR?

On general grounds, the anomaly is understood as a **fundamental incompatibility** between the classical symmetry and the regularization procedure.

The symmetry is anomalous because the breaking introduced by the regularization **cannot** be subtracted by a **local counterterm** added to the action.

From this point of view the anomaly can be regarded as a **UV effect**.

But there is **also an IR side**…

Let us look at the **on-shell amplitude**

$$\langle 0|J_A^\mu(0)|p,q\rangle_{\mathscr{A}} = \Gamma^{\mu\alpha\beta}(p,q)\widetilde{\mathscr{A}_\alpha}(p)\widetilde{\mathscr{A}_\beta}(q)\Big|_{p^2=q^2=0}$$

where $p^\mu\widetilde{\mathscr{A}_\mu}(p) = 0$. We recall,

$$\begin{aligned} i\Gamma_{\mu\alpha\beta}(p,q) &= f_1\epsilon_{\mu\alpha\beta\sigma}p^\sigma + f_2\epsilon_{\mu\alpha\beta\sigma}q^\sigma + f_3\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda \\ &+ f_4\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda + f_5\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda + f_6\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda \end{aligned}$$

and due to the on-shell condition

$$f_i(p,q) = f_i(p\cdot q)$$ (symmetric in p and q)

and from Bose symmetry $f_1 = -f_2$, $f_3 = -f_6$, and $f_4 = -f_5$.

Vector current conservation further implies:

$$f_2 - p^2 f_5 - p\cdot q f_6 = 0$$
$$f_1 - q^2 f_4 - p\cdot q f_3 = 0$$

→

$$f_1(p,q) = p\cdot q f_3(p,q)$$

The amplitude is then given only in terms of $f_3(p,q)$ and $f_4(p,q)$

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q)\Big|_{p^2=q^2=0} &= f_3(p,q)\Big[p\cdot q\,\epsilon_{\mu\alpha\beta\sigma}(p^\sigma-q^\sigma)+\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda-\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda\Big] \\
&+ f_4(p,q)\Big(\epsilon_{\mu\alpha\sigma\lambda}q_\beta-\epsilon_{\mu\beta\sigma\lambda}p_\alpha\Big)p^\sigma q^\lambda
\end{aligned}
$$

Due to $p^\mu\widetilde{\mathscr{A}_\mu}(p)=0$, the term with $f_4(p,q)$ does not contribute to the amplitude.

Using as well

$$
\begin{aligned}
-p\cdot q\epsilon_{\mu\alpha\beta\sigma}p^\sigma &= \epsilon_{\alpha\beta\sigma\lambda}p_\mu p^\sigma q^\lambda+\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda+\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda, \\
p\cdot q\epsilon_{\mu\alpha\beta\sigma}q^\sigma &= \epsilon_{\alpha\beta\sigma\lambda}q_\mu p^\sigma q^\lambda+\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda+\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda.
\end{aligned}
$$

the amplitude takes the form:

$$
\langle 0|J^\mu_{\rm A}(0)|p,q\rangle_{\mathscr{A}}=i(p+q)^\mu f_3(p,q)\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda\widetilde{\mathscr{A}^\alpha}(p)\widetilde{\mathscr{A}^\beta}(q)
$$

The amplitude is then given only in terms of $f_3(p,q)$ and $f_4(p,q)$

$$
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i\Gamma_{\mu\alpha\beta}(p,q)\bigg|_{p^2=q^2=0} &= f_3(p,q)\Big[p\cdot q\,\epsilon_{\mu\alpha\beta\sigma}(p^\sigma-q^\sigma)+\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda-\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda\Big] \\
&+ f_4(p,q)\Big(\epsilon_{\mu\alpha\sigma\lambda}q_\beta-\epsilon_{\mu\beta\sigma\lambda}p_\alpha\Big)p^\sigma q^\lambda
\end{aligned}
$$

Due to $p^\mu\widetilde{\mathscr{A}_\mu}(p)=0$, the term with $f_4(p,q)$ does not contribute to the amplitude.

Using as well

$$\epsilon_{\alpha\beta\sigma\lambda}w_\mu+\epsilon_{\beta\sigma\lambda\mu}w_\alpha+\epsilon_{\sigma\lambda\mu\alpha}w_\beta+\epsilon_{\lambda\mu\alpha\beta}w_\sigma+\epsilon_{\mu\alpha\beta\sigma}w_\lambda=0$$

$$
\begin{aligned}
-p\cdot q\epsilon_{\mu\alpha\beta\sigma}p^\sigma &= \epsilon_{\alpha\beta\sigma\lambda}p_\mu p^\sigma q^\lambda+\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda+\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda, \\
p\cdot q\epsilon_{\mu\alpha\beta\sigma}q^\sigma &= \epsilon_{\alpha\beta\sigma\lambda}q_\mu p^\sigma q^\lambda+\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda+\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda.
\end{aligned}
$$

the amplitude takes the form:

$$\langle 0|J^\mu_{\rm A}(0)|p,q\rangle_{\mathscr{A}}=i(p+q)^\mu f_3(p,q)\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda\widetilde{\mathscr{A}^\alpha}(p)\widetilde{\mathscr{A}^\beta}(q)$$

$$\langle 0|J^\mu_{\rm A}(0)|p,q\rangle_{\mathscr{A}} = i(p+q)^\mu f_3(p,q)\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda \widetilde{\mathscr{A}}^\alpha(p)\widetilde{\mathscr{A}}^\beta(q)$$

The function $f_3(p,q)$ can be computed from Feynman diagrams

$$f_3(p,q) = \frac{ie^2}{\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{xy}{2xyp\cdot q - m^2}$$

If we take a **naive** massless limit,

$$\lim_{m\to 0} f_3(p,q) = \frac{ie^2}{2\pi^2}\frac{1}{(p+q)^2}$$

and we find

$$\lim_{m\to 0}\langle 0|J^\mu_A(0)|p,q\rangle_{\mathscr{A}} = -\frac{e^2}{2\pi^2}\frac{(p+q)^\mu}{(p+q)^2}\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda \widetilde{\mathscr{A}}^\alpha(p)\widetilde{\mathscr{A}}^\beta(q).$$

At the level of the **current**, the anomaly is signalled by a **massless pole**!

Thus, the anomaly has two faces:

- When looking at the **divergence of the current**, it comes associated with ambiguities in the **UV** behavior of the integrals. Fixing them forces us to give up the axial-vector symmetry in favor of gauge invariance.

- When looking at the **current itself**, it is signaled by the appearance of a **massless pole** (i.e., an **IR effect**)

In fact, being careful, we should have written the result for the amplitude as

$$\lim_{m\to 0}\langle 0|J_A^\mu(0)|p,q\rangle_{\mathscr{A}} = -\frac{e^2}{2\pi^2}\frac{(p+q)^\mu}{(p+q)^2+i\epsilon}\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda \widetilde{\mathscr{A}}^\alpha(p)\widetilde{\mathscr{A}}^\beta(q).$$

The reason is that the integration over y in

$$f_3(p,q) = \frac{ie^2}{\pi^2}\int_0^1 dx\int_0^{1-x} dy\frac{xy}{2xyp\cdot q - m^2}$$

produces a logarithm and an imaginary part

$$\mathrm{Im}\, f_3(p,q)\neq 0 \qquad \text{for} \qquad (p+q)^2 > 4m^2$$

when $m\to 0$ the real part develops a pole and the imaginary part a delta-function singularity **whose coefficient is the anomaly**

$$\lim_{m\to 0}\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q) = \frac{e^2}{2\pi}\epsilon^{\alpha\beta\sigma\lambda}p_\sigma q_\lambda(p+q)^\mu\delta\Big((p+q)^2\Big)$$

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produces a logarithm and an imaginary part

$\mathrm{Im}\, f_3(p,q) \neq 0$ for $(p+$

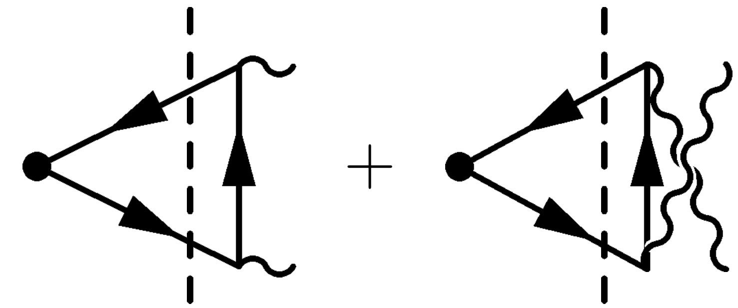
$$\frac{1}{x+i\epsilon} = \mathrm{PV}\frac{1}{x} - i\pi\delta(x)$$

when $m\to 0$ the real part develops a pole and the imaginary part a delta-function singularity **whose coefficient is the anomaly**

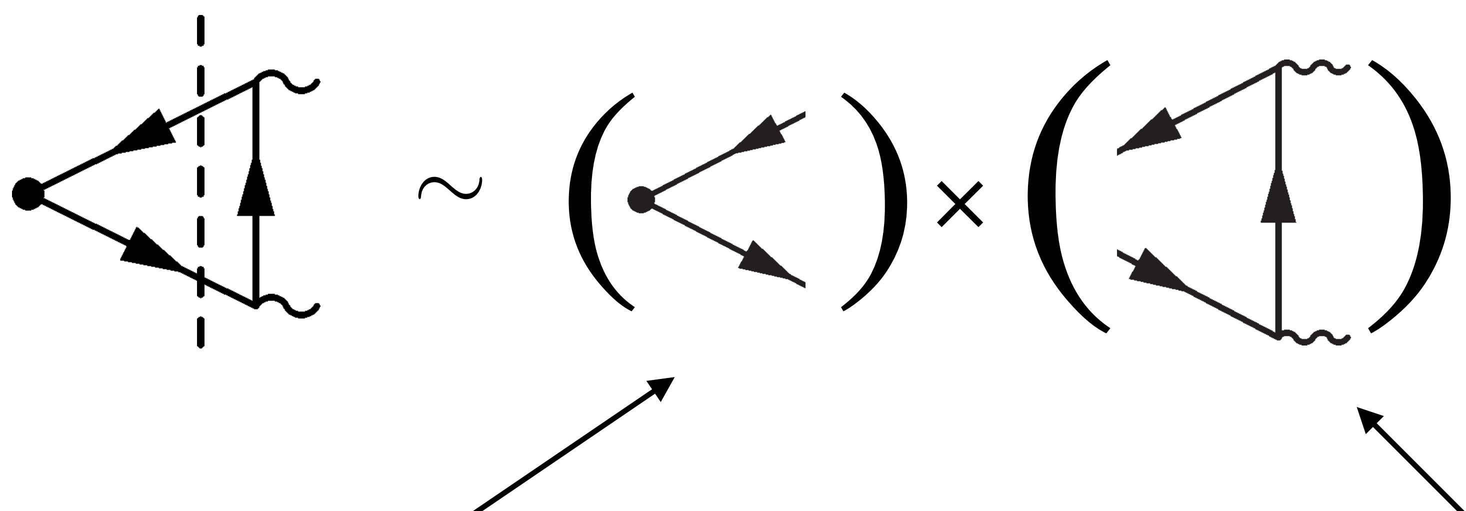
$$\lim_{m\to 0}\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q) = \frac{e^2}{2\pi}\epsilon^{\alpha\beta\sigma\lambda}p_\sigma q_\lambda(p+q)^\mu\delta\Big((p+q)^2\Big)$$

This discontinuity in the imaginary part of the amplitude can be understood **physically**.

Let us use the **Cutkosky rules**:

$$\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q) \sim$$ 

where, e.g.



fermion-antifermion creation by J^{μ}_{A}

fermion-antifermion annihilation (2 diagrams)

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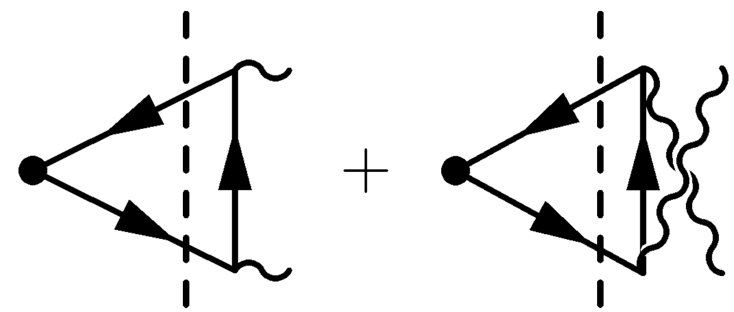
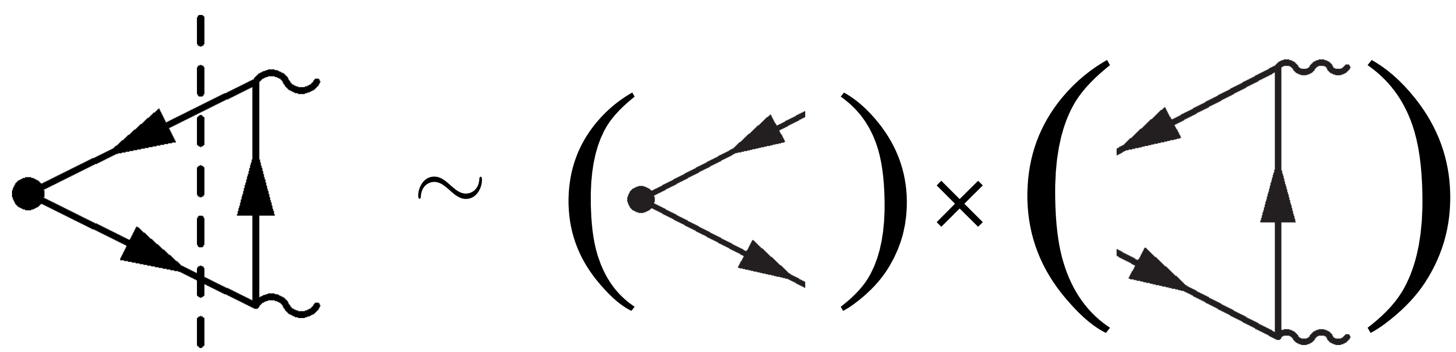
Let us use the **Cutkosky rules**:

$$\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q) \sim$$



where, e.g.



$$\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q)\epsilon_\alpha(\mathbf{p},\lambda_1)\epsilon_\beta(\mathbf{q},\lambda_2) \sim \sum_{\sigma_1,\sigma_2}\int d^3k_1\int d^3k_2\ {}_{\rm out}\langle \mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma_1;\mathbf{k}_2,\sigma_2\rangle_{\rm in}$$

$$\times\,{}_{\rm out}\langle \mathbf{k}_1,\sigma_1;\mathbf{k}_2,\sigma_2|J_A^\mu(0)|0\rangle_{\rm in}$$

$$\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q)\epsilon_\alpha(\mathbf{p},\lambda_1)\epsilon_\beta(\mathbf{q},\lambda_2) \sim \sum_{\sigma_1,\sigma_2}\int d^3k_1\int d^3k_2\; {}_{\rm out}\langle \mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma_1;\mathbf{k}_2,\sigma_2\rangle_{\rm in}$$

$$\times\, {}_{\rm out}\langle \mathbf{k}_1,\sigma_1;\mathbf{k}_2,\sigma_2|J^\mu_A(0)|0\rangle_{\rm in}$$

The first important thing is to invoke the **Landau-Yang theorem**: no state of spin-one can decay into two on-shell photons.

Thus, the fermion-antifermion system should have **zero spin**. This means that in the center of mass frame they have the **same helicities**

$$\sigma_1 = \sigma_2 \equiv \sigma$$

We begin with the pair creation by the axial-vector current:

$$ {}_{\rm out}\langle \mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma|J^\mu_A(0)|0\rangle_{\rm in} \sim \overline{v}(\mathbf{k}_1,\sigma)\gamma^\mu\gamma_5 u(\mathbf{k}_2,\sigma) $$

In the massless limit, the helicity turns into ± chirality

$$ \lim_{m\to 0} u(\mathbf{p},\pm\frac{1}{2}) = u_\pm(\mathbf{p}) \qquad\qquad \lim_{m\to 0} v(\mathbf{p},\pm\frac{1}{2}) = v_\mp(\mathbf{p}) $$

Thus, using

$$ \overline{v}_\mp(\mathbf{k}_2)\gamma^\mu\gamma_5 u_\pm(\mathbf{k}_1) = 0 $$

we find

$$ \lim_{m\to 0} {}_{\rm out}\langle \mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma|J^\mu_A(0)|0\rangle_{\rm in} = 0 $$

We turn now to the annihilation of the two fermions

$$
{}_{\rm out}\langle \mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma\rangle_{\rm in} = -e^2\epsilon_\mu(\mathbf{p},\lambda_1)\epsilon_\nu(\mathbf{k},\lambda_2)
$$

$$
\times\overline{v}(\mathbf{k}_2,\sigma)\left[\frac{\gamma^\mu(\not{k}_1-\not{p}+m)\gamma^\nu}{(k_1-p)^2-m^2}+\frac{\gamma^\nu(\not{k}_2-\not{q}+m)\gamma^\mu}{(k_2-q)^2-m^2}\right]u(\mathbf{k}_1,\sigma)
$$

Using now that

$$
\overline{v}_\mp(\mathbf{k}_2)\gamma^\mu\gamma^\alpha\gamma^\nu u_\pm(\mathbf{k}) = 0.
$$

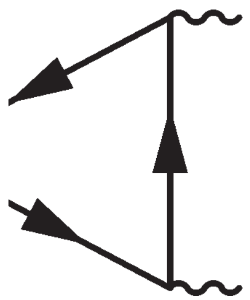
we conclude that the second amplitude also vanish in the massless limit

$$
\lim_{m\to 0}\,{}_{\rm out}\langle \mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma\rangle_{\rm in} = 0
$$

Thus, we would find that the amplitude approaches zero with the mass

$$
\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q)\sim 0
$$

$$
{}_{\rm out}\langle \mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma\rangle_{\rm in} = -e^2\epsilon_\mu(\mathbf{p},\lambda_1)\epsilon_\nu(\mathbf{k},\lambda_2)
$$
$$
\times\overline{v}(\mathbf{k}_2,\sigma)\left[\frac{\gamma^\mu(\not{k}_1-\not{p}+m)\gamma^\nu}{(k_1-p)^2-m^2}+\frac{\gamma^\nu(\not{k}_2-\not{q}+m)\gamma^\mu}{(k_2-q)^2-m^2}\right]u(\mathbf{k}_1,\sigma)
$$



But not so fast...

In the massless limit, on-shell fermions can emit collinear on-shell photons, and the intermediate state can fall on-shell.

The denominator then vanishes and we have an indeterminate limit.

That is why, being more careful we obtained:

$$
\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q)\sim(\text{anomaly})\times\delta\Big((p+q)^2\Big)
$$

Thus, the anomaly appears as an **IR discontinuity** of the imaginary part of the amplitude.

Interestingly, this imaginary part in **unambiguous**.

A two-dimensional excursion: the Schwinger model

To keep things simple, we consider a **massless** Dirac fermion in 1+1 dimensions, and **compactify** the spatial direction to a circle of length L.

We consider the following representation of the Dirac matrices

$$\gamma^0 \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with the chirality matrix given by

$$\gamma_5 \equiv -\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Decomposing the Dirac fermion into its Weyl components $\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ the Dirac equation reads

$$(\partial_0 - \partial_1)u_+ = 0, \qquad (\partial_0 + \partial_1)u_- = 0$$

$$u_+ = u_+\underbrace{(x^0 + x^1)}_{\text{left-mover}}, \qquad u_- = u_-\underbrace{(x^0 - x^1)}_{\text{right-mover}}$$

chirality is linked to the **direction of motion**

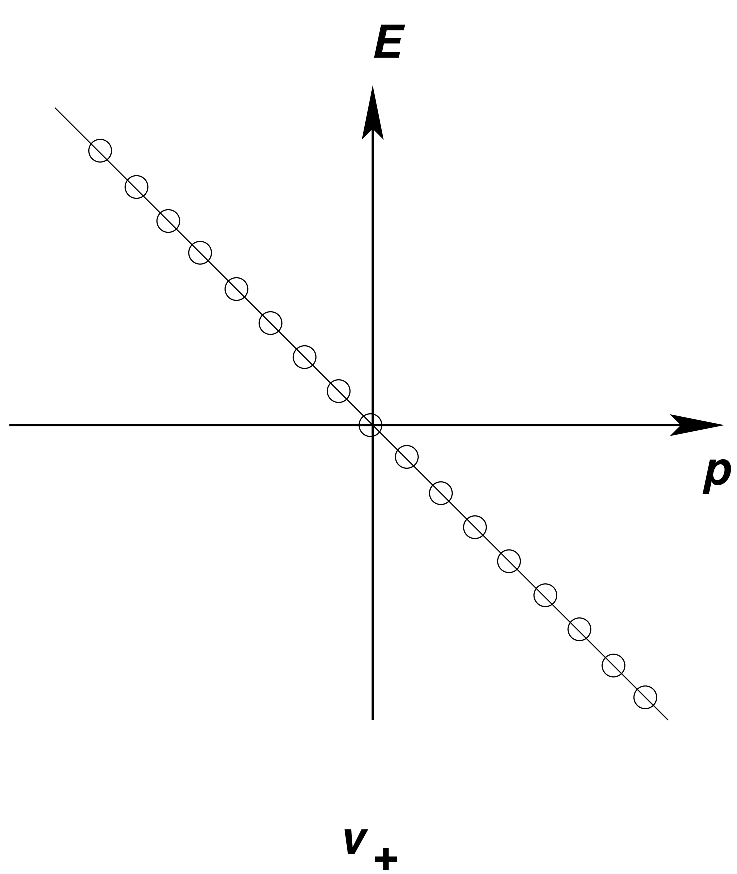
the wave function for free fermions are

$$v_{\pm}^{(E)}(x^0 \pm x^1) = \frac{1}{\sqrt{L}} e^{-iE(x^0 \pm x^1)} \qquad \text{with} \qquad E = \mp p$$

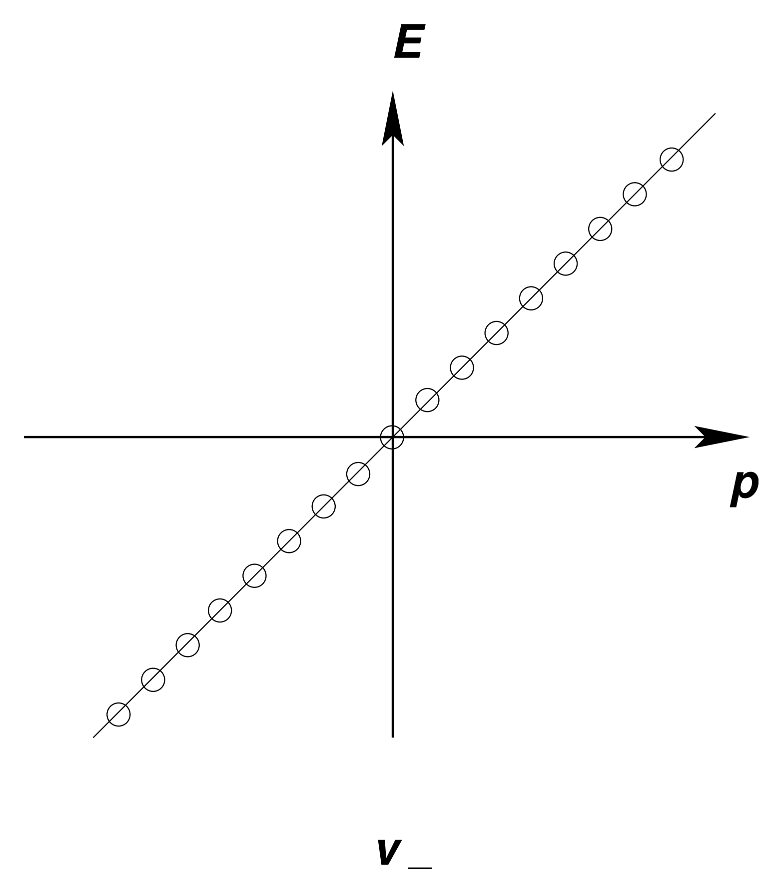
and since the spatial direction is compatified, the momentum is **quantized**:

$$p = \frac{2\pi n}{L}, \qquad n \in \mathbb{Z}$$

the **spectrum** is:

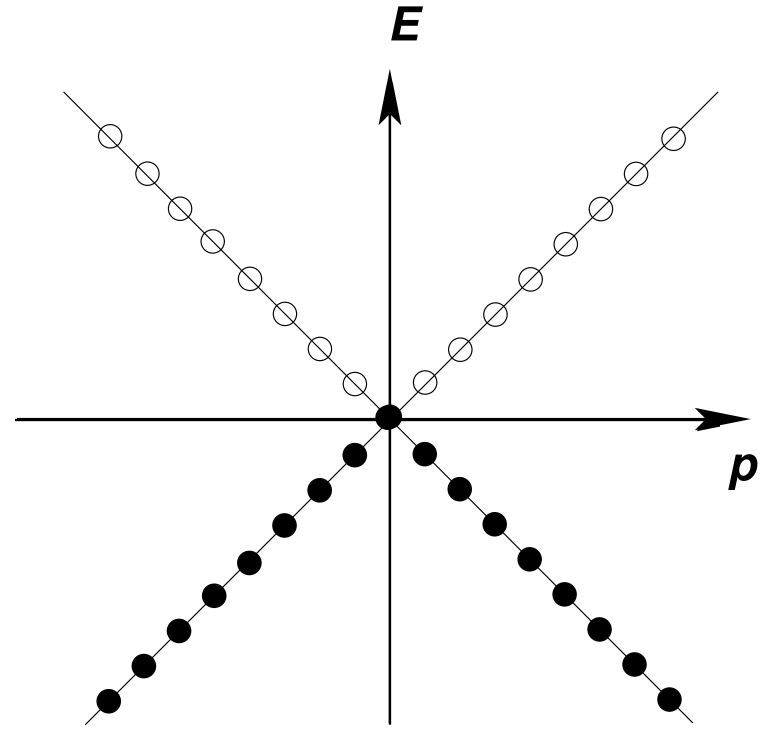


(positive chirality, left movers)



(negative chirality, right movers)

To quantize the Dirac fermion, we construct firts the **ground state** of the theory by filling all negative energy states (Dirac sea)



and expand:

$$u_{\pm}(x) = \sum_{E>0} \left[a_{\pm}(E) v_{\pm}^{(E)}(x) + b_{\pm}^{\dagger}(E) v_{\pm}^{(E)}(x)^* \right]$$

where,

$a_{\pm}(E)$: annihilates a **fermion** with $E>0$ and $p=\mp E$

$b_{\pm}^{\dagger}(E)$: creates an **antifermion** with $E>0$ and $p=\pm E$ → ($\pm$ chirality)

(i.e., annihilates a fermion with $E<0$ and $p=\mp E$)

We look now at the **classical symmetries** of our theory

$$\mathcal{L} = iu_+^\dagger(\partial_0 + \partial_1)u_+ + iu_-^\dagger(\partial_0 - \partial_1)u_-$$

Vector U(1):

$$\psi \longrightarrow e^{i\alpha}\psi$$

$$u_\pm \longrightarrow e^{i\alpha}u_\pm$$

whose associated Noether current is

$$J^\mu_{\rm V} = \overline{\psi}\gamma^\mu\psi$$

$$J^\mu_{\rm V} = \left(\begin{array}{c} u_+^\dagger u_+ + u_-^\dagger u_- \\ -u_+^\dagger u_+ + u_-^\dagger u_- \end{array}\right)$$

Axial U(1):

$$\psi \longrightarrow e^{i\beta\gamma_5}\psi$$

$$u_\pm \longrightarrow e^{\pm i\beta}u_\pm$$

with

$$J^\mu_{\rm A} = \overline{\psi}\gamma^\mu\gamma_5\psi$$

$$J^\mu_{\rm A} = \left(\begin{array}{c} u_+^\dagger u_+ - u_-^\dagger u_- \\ -u_+^\dagger u_+ - u_-^\dagger u_- \end{array}\right)$$

the corresponding conserved charges are

$$Q_{\rm V} \equiv \int_0^L dx^1\, J^0_{\rm V} = \int_0^L dx^1 \left(u_+^\dagger u_+ + u_-^\dagger u_-\right)$$

$$Q_{\rm A} \equiv \int_0^L dx^1\, J^0_{\rm A} = \int_0^L dx^1 \left(u_+^\dagger u_+ - u_-^\dagger u_-\right)$$

Using the orthogonality relations of the wave functions

$$\int_0^L dx^1\, v_\pm^{(E)}(x)^*\, v_\pm^{(E')}(x) = \delta_{E,E'}$$

$$u_\pm(x) = \sum_{E>0}\left[a_\pm(E)v_\pm^{(E)}(x) + b_\pm^\dagger(E)v_\pm^{(E)}(x)^*\right]$$

we find

$$Q_{\rm V} \;=\; \sum_{E>0}\left[a_+^\dagger(E)a_+(E) - b_+^\dagger(E)b_+(E) + a_-^\dagger(E)a_-(E) - b_-^\dagger(E)b_-(E)\right]$$

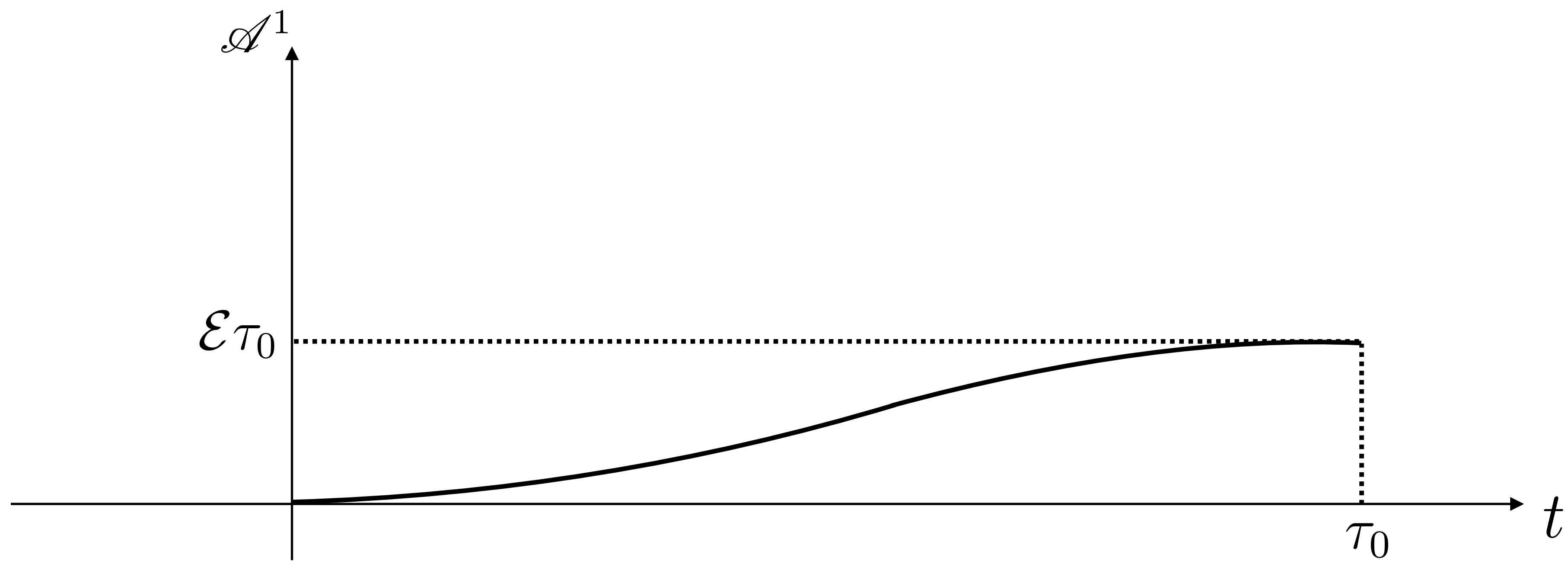
(fermions minus antifermions)

$$Q_{\rm A} \;=\; \sum_{E>0}\left[a_+^\dagger(E)a_+(E) - b_+^\dagger(E)b_+(E) - a_-^\dagger(E)a_-(E) + b_-^\dagger(E)b_-(E)\right]$$

("**net**" number of +'ve minus −'ve chirality states)

In the free theory, both charges are conserved… but what about switching an **external electrical field**?

We do it adiabatically. In the $\mathscr{A}^0 = 0$ gauge



The effect of this external field on the fermions is shifting the momentum by

$$p \longrightarrow p - e\mathscr{A}^1$$

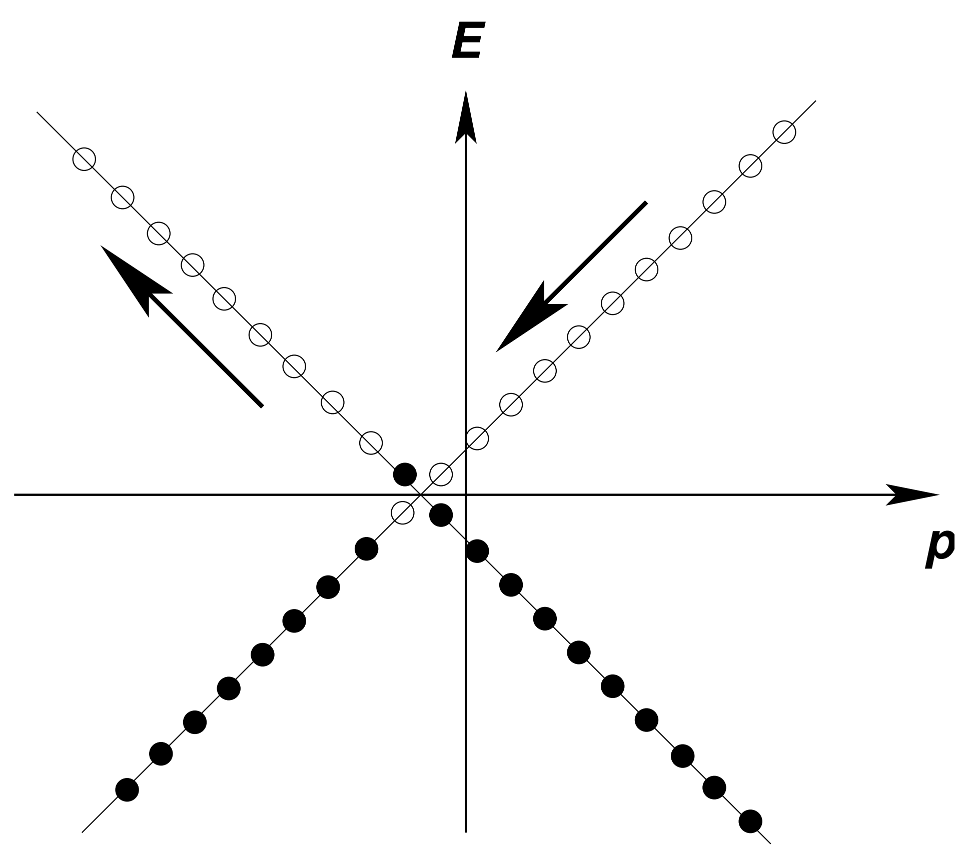
Adiabaticity allows to treat the system at each instant as "time independent" (no transitions).

$$p \longrightarrow p - e\mathscr{A}^1$$

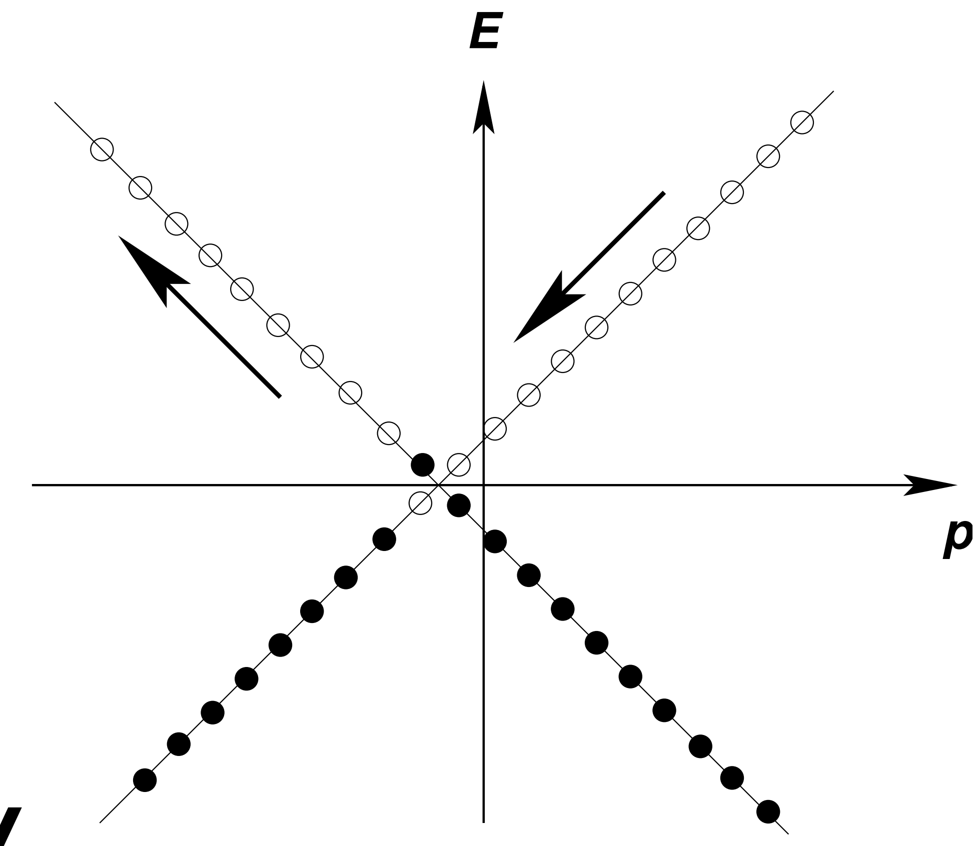
The shift have different effects on the states on each branch of the spectrum:

$$E = p \quad \longrightarrow \quad E = p - e\mathscr{A}^1 \qquad \text{(it "sinks")}$$

$$E = -p \quad \longrightarrow \quad E = -p + e\mathscr{A}^1 \qquad \text{(it "raises")}$$



A number of negative chirality empty states become “holes” (negative chirality antifermions), while some occupied negative energy states with positive chirality get positive energy (positive chirality fermions)



The external field creates **pairs** of **+’ve chirality fermions** and **−‘ve chirality antifermions**!

But, how many pairs?

$$N = \frac{\text{shift in the spectrum}}{\text{spectrum gap}} = \frac{e\mathscr{E}\tau_0}{2\pi/L} \longrightarrow N = \frac{L}{2\pi}e\mathscr{E}\tau_0$$

This preserves the vector charge:

$$Q_{\rm V}(\tau_0) = (N-0)+(0-N)=0$$

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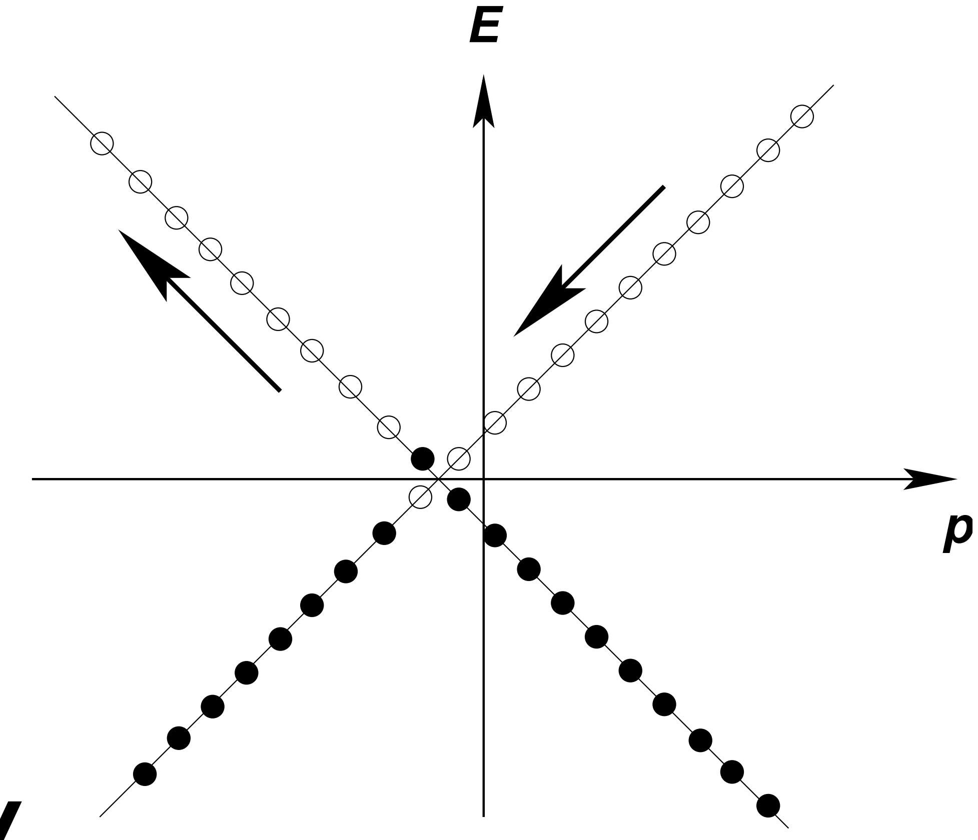

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But changes the axial charge:

$$Q_{\mathrm{A}}(\tau_0) \;=\; (N-0)-(0-N) = 2N$$

$$Q_{\rm A}(\tau_0) \;=\; (N-0)-(0-N)=2N \qquad\qquad N=\frac{L}{2\pi}e\mathscr{E}\tau_0$$

We have found that, in the presence of an external electric field, there is a **violation in the conservation of the axial current**.

Its rate of variation is

$$\dot{Q}_A=\frac{Q_A(\tau_0)}{\tau_0}=\frac{e}{\pi}L\mathscr{E}$$

This implies a violation in the conservation of the axial current

$$\partial_\mu J^\mu_{\rm A}=\frac{e}{\pi}\mathscr{E}$$

which gives the value of the axial anomaly in the Schwinger model:

$$\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}}=\frac{e}{2\pi}\epsilon^{\mu\nu}\mathscr{F}_{\mu\nu}(x)$$

The anomaly in the massless Schwinger model has **surprising consequences**...

In fact, in two dimensions the vector and axial-vector currents are closely related.

$$\gamma_5 = -\gamma^0\gamma^1 \quad \longrightarrow \quad \gamma^\mu\gamma_5 = \epsilon^{\mu\nu}\gamma_\nu$$

Hence,

$$J^\mu_{\mathrm{A}}(x) = \epsilon^{\mu\nu} J_{\mathrm{V}\mu}(x)$$

Thus the anomaly can be recast in terms of the vector current as

$$\epsilon^{\mu\nu}\partial_\mu\langle J_{\mathrm{V}\nu}(x)\rangle_{\mathscr{A}} = \frac{e}{2\pi}\epsilon^{\mu\nu}\mathscr{F}_{\mu\nu}(x) = \frac{e}{\pi}\epsilon^{\mu\nu}\partial_\mu\mathscr{A}_\nu(x)$$

$$\epsilon^{\mu\nu}\partial_\mu\langle J_{\mathrm{V}\nu}(x)\rangle_{\mathscr{A}} = \frac{e}{2\pi}\epsilon^{\mu\nu}\mathscr{F}_{\mu\nu}(x) = \frac{e}{\pi}\epsilon^{\mu\nu}\partial_\mu\mathscr{A}_\nu(x)$$

In addition, the vector current has to satisfy the Maxwell equations

$$\partial_\mu\mathscr{F}^{\mu\nu}(x) = -e\langle J_{\mathrm{V}}^\nu(x)\rangle_{\mathscr{A}} \quad \longrightarrow \quad \Box\mathscr{A}^\nu(x) - \partial^\nu\partial_\mu\mathscr{A}^\mu(x) = -e\langle J_{\mathrm{V}}^\nu(x)\rangle_{\mathscr{A}}$$

Defining the pseudoscalar field $\mathscr{F}^* \equiv \frac{1}{2}\epsilon_{\mu\nu}\mathscr{F}^{\mu\nu} = \epsilon^{\mu\nu}\partial_\mu\mathscr{A}_\nu$ the two equations combine into:

$$\left(\Box + \frac{e^2}{\pi}\right)\mathscr{F}^* = 0$$

This means that the Schwinger model contains a propagating mode with mass

$$m^2 = \frac{e^2}{\pi}$$

What is this mode? Let's remember than in two dimensions, a vector can be decomposed as

$$A_\mu = \partial_\mu \eta + \epsilon_{\mu\nu} \partial^\nu \eta'$$

Due to the interaction with the fermions, the pseudoscalar mode acquires a mass.

The 2D Dirac fermion works like a "**technifermion**" which produces a **massive photon**.

Unfortunately, this only works in 2D!

What is this mode? Let's remember than in two dimensions, a vector can be decomposed as

$$A_\mu = \partial_\mu \eta + \epsilon_{\mu\nu}\partial^\nu \eta'$$

(pure gauge: $\partial_\mu \eta$; pseudoscalar: η')

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The 2D Dirac fermion works like a "**technifermion**" which produces a **massive photon**.

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The Dirac-sea picture of the anomaly in the Schwinger model underlines its **IR character**

The anomaly it is determined by a number of states crossing the $E=0$ Fermi level