Gravitational anomalies
(first contact)

Gravitons are quantized perturbations over flat (or any other background) spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \qquad \left(\kappa = \sqrt{8\pi G_N} \right)$$

The graviton action is obtained expanding the **Einstein-Hilbert action** around the Minkowski metric

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R[g]$$

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \frac{1}{2}\partial^\alpha h_{\alpha\beta}\partial_\mu h^{\mu\beta} + \text{self-interactions}\right)$$

At the level of the graviton field, diffeomorphism invariance translate into gauge tranformations generated by a vector field

$$\delta h_{\mu\nu}(x) = \frac{1}{2}\Big[\partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x)\Big]$$

Expanding the matter action to linear order in the graviton field

$$S[\phi_i,\eta+2\kappa h]=S[\phi_i]+2\kappa\left(\int d^4x\,h_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}\right)\bigg|_{g=\eta}$$

$$=S[\phi_i]-\kappa\left[\int d^4x\sqrt{-g}h_{\mu\nu}\left(-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\right)\right]\bigg|_{g=\eta}$$

leads to the coupling between the graviton and the energy-momentum tensor

$$S_{\rm int}=-\kappa\int d^4x\,h_{\mu\nu}T^{\mu\nu}$$

Invariance under gauge transformations

$$\delta h_{\mu\nu}(x)=\frac{1}{2}\Big[\partial_\mu\xi_\nu(x)+\partial_\nu\xi_\mu(x)\Big]$$

depends on the **conservation of the energy-momentum tensor**

$$\delta S_{\rm int}=\kappa\int d^4x\,\xi_\nu\partial_\mu T^{\mu\nu} \quad \longrightarrow \quad \partial_\mu T^{\mu\nu}=0$$

Gravitational anomalies appear whenever the **energy-momentum tensor is not conserved quantum-mechanically**

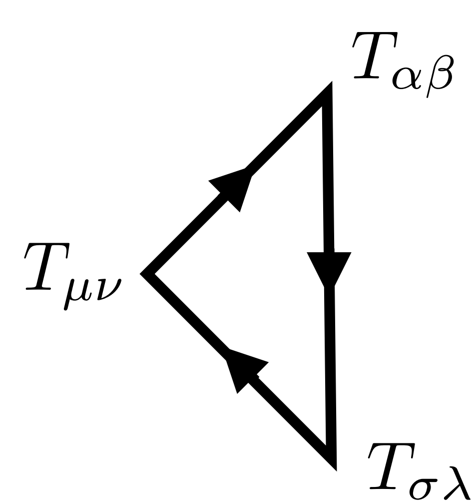
$$\partial_\mu \langle T^{\mu\nu}(x)\rangle_h \neq 0$$

Let us consider a theory of a chiral fermion coupled to a background graviton field

$$T_{\mu\nu} = \frac{i}{4}\overline{\psi}\Big(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu\Big)\psi \qquad \text{where} \qquad f_1 \overleftrightarrow{\partial}_\nu f_2 = f_1(\partial_\mu f_2) - (\partial_\mu f_1) f_2$$

The expectation value of the energy-momentum tensor is then

$$\langle T^{\mu\nu}(x)\rangle_h = \frac{1}{Z}\int \mathscr{D}\psi\mathscr{D}\overline{\psi}\, T^{\mu\nu}(x)\, e^{i\int d^4y(i\overline{\psi}_+\gamma^\mu\partial_\mu\psi_+ - \kappa h^{\mu\nu}T_{\mu\nu})}$$



Expanding in powers of κ we find again the **triangle diagram**, this time with three energy-momentum tensor insertions

But, since anomalies and parity noninvariance come together, the question is whether the coupling to gravity is sensitive to chirality

This depends on the dimension:

- $D = 4k$:

CPT **reverses** the helicity of fermions

- $D = 4k+2$:

CPT **preserves** the helicity of fermions

Thus, in $D = 4k$ there are as many left-handed as right-handed fermions

+ “equivalence principle”

There are no pure gravitational anomalies in four dimensions

However, gravity can **contribute** to the **gauge anomaly**…

For example, a left-handed fermion coupled to a gauge field also couples to gravity through

$$S=\int d^4x\left[i\overline{\psi}\gamma^\mu\partial_\mu\psi+\overline{\psi}\gamma^\mu T^a\left(\frac{1-\gamma_5}{2}\right)\psi\mathscr{A}_\mu^a-\kappa h_{\mu\nu}T^{\mu\nu}\right]$$

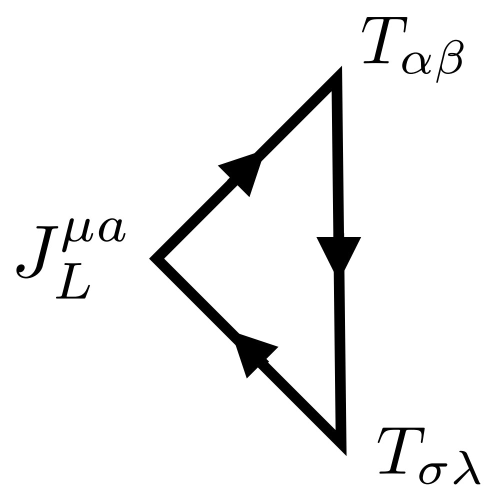
The quantum conservation of the current is then

$$\langle J_L^{\mu a}(x)\rangle_{\mathscr{A},h}=\frac{1}{Z}\int\mathscr{D}\psi\mathscr{D}\overline{\psi}J_L^{\mu a}(x)e^{iS_0[\psi,\overline{\psi},\mathscr{A}]-i\kappa\int d^4y\,T^{\mu\nu}h_{\mu\nu}}$$

Doing perturbation theory in powers of κ brings down insertions of the energy-momentum tensor into the correlation function. Then, we have contributions like

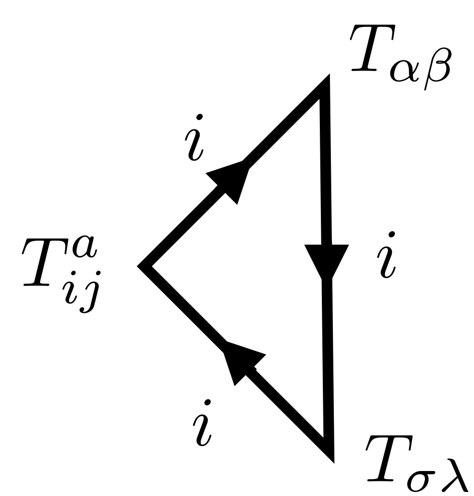
$$-\frac{\kappa^2}{2}\int d^4y_1\int d^4y_2\,\langle 0|T\left[J_L^{\mu a}(x)T^{\alpha\beta}(y_1)T^{\sigma\lambda}(y_2)\right]|0\rangle h_{\alpha\beta}(y_1)h_{\sigma\lambda}(y_2)$$

Diagrammatically, we have again a triangle diagram with a gauge current and two energy-momentum tensors



+Bose symmetry

Since we are only interested in cancelling this contribution we just need to look at the group theory factor



$\sim \ \mathrm{Tr}\, T^a$

Thus, the condition for the cancellation of **mixed gauge-gravitational anomalies** is

$$\sum_{\text{right-handed}} \operatorname{Tr} T^a_+ - \sum_{\text{left-handed}} \operatorname{Tr} T^a_- = 0$$

- $\mathrm{SU}(N)$ for $N \geq 2$ do not contribute to mixed anomalies (tracelessness!)
- But **beware** of $\mathrm{U}(1)$'s!!!

The cancellation of mixed anomalies poses very **strong nontrivial constraint** on theories (e.g. the standard model).

Phenomenology of anomalies I: Pion decay

Warming up: the Goldstone theorem

We consider a QFT with a global symmetry and an associated conserved current

$$\partial_\mu J^\mu = 0 \quad \longrightarrow \quad \begin{cases} Q(t) = \displaystyle\int d^3x\, J^0(x) \\ \dot{Q}(t) = 0 \end{cases}$$

Given now an observable $\mathscr{O}(x)$ we compute

$$\begin{aligned}\langle\Omega|[Q(t),\mathscr{O}(0)]|\Omega\rangle &= \int d^3x\,\langle\Omega|[J^0(t,\mathbf{x}),\mathscr{O}(0)]\Omega\rangle \\ &= \int d^3x\Big[\langle\Omega|J^0(t,\mathbf{x})\mathscr{O}(0)|\Omega\rangle - \langle\Omega|\mathscr{O}(0)J^0(t,\mathbf{x})|\Omega\rangle\Big]\end{aligned}$$

Inserting now a basis $|n\rangle$ of energy eigenstates

$$\begin{aligned}\langle\Omega|[Q(t),\mathscr{O}(0)]|\Omega\rangle &= \sum_n\int d^3x\,\Big[\langle\Omega|J^0(x)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle - \langle\Omega|\mathscr{O}(0)|n\rangle\,\langle\Omega|J^0(x)|\Omega\rangle\Big] \\ &= \sum_n\int d^3x\,\Big[e^{-iP_n\cdot x}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle - e^{iP_n\cdot x}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\Big]\end{aligned}$$

Warming up: the Goldstone theorem

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Given now an observable $\mathscr{O}(x)$ we compute

$$J^0(x) = e^{iP\cdot x} J^0(0) e^{-iP\cdot x}$$

$$\begin{aligned} \langle\Omega|[Q(t),\mathscr{O}(0)]|\Omega\rangle &= \int d^3x\,\langle\Omega|[J^0(t,\mathbf{x}),\mathscr{O}(0)]\Omega\rangle \\ &= \int d^3x\Big[\langle\Omega|J^0(t,\mathbf{x})\mathscr{O}(0)|\Omega\rangle - \langle\Omega|\mathscr{O}(0)J^0(t,\mathbf{x})|\Omega\rangle\Big] \end{aligned}$$

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$$\langle\Omega|[Q(t),\mathscr{O}(0)]|\Omega\rangle=\sum_n\int d^3x\left[e^{-iP_n\cdot x}\langle\Omega|J^0(0)|n\rangle\langle n|\mathscr{O}(0)|\Omega\rangle-e^{iP_n\cdot x}\langle\Omega|\mathscr{O}(0)|n\rangle\langle n|J^0(0)|\Omega\rangle\right]$$

and carrying out the integral,

$$\begin{aligned}\langle\Omega|[Q(t),\mathscr{O}(0)]|\Omega\rangle=\sum_n(2\pi)^3\delta^{(3)}(\mathbf{P}_n)\Big[&e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\langle n|\mathscr{O}(0)|\Omega\rangle\\&-e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\langle n|J^0(0)|\Omega\rangle\Big]\end{aligned}$$

Now, if the symmetry is spontaneously broken $Q(t)|\Omega\rangle\neq 0$ and we have

$$\sum_n\delta^{(3)}(\mathbf{P}_n)\Big[e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\langle n|\mathscr{O}(0)|\Omega\rangle-e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\langle n|J^0(0)|\Omega\rangle\Big]\neq 0$$

and since $\dot{Q}(t)=0$, taking the time derivative we arrive at

$$\sum_n E_n\delta^{(3)}(\mathbf{P}_n)\Big[e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\langle n|\mathscr{O}(0)|\Omega\rangle+e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\langle n|J^0(0)|\Omega\rangle\Big]=0$$

$$\sum_n \delta^{(3)}(\mathbf{P}_n)\Big[e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle - e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\Big] \neq 0$$

$$\sum_n E_n\delta^{(3)}(\mathbf{P}_n)\Big[e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle + e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\Big] = 0$$

This two equations imply the existence of a state $|m\rangle$ such that

$$\langle\Omega|J^0(0)|m\rangle \neq 0$$

$$E_m\delta^{(3)}(\mathbf{P}_m) = 0 \quad \longrightarrow \quad E_m(\mathbf{P}_m = \mathbf{0}) = 0$$

This state is the **Goldstone boson**

- It is massless
- It is created by the current from the vacuum $\langle m|J^\mu(x)|\Omega\rangle \neq 0$
- It has the same quantum numbers as the conserved current

Application to chiral symmetry breaking

Massless QCD:

$$S_{\chi\mathrm{QCD}} = \int d^4x \left[-\frac{1}{4}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \sum_{f=1}^{N_F} \left(i\overline{Q}^f_L \not{D} Q^f_L + i\overline{Q}^f_R \not{D} Q^f_R \right) \right]$$

is classically invariant under $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ **global chiral** transformations of the quark fields

$$\mathrm{U}(N_f)_L : \begin{cases} Q^f_L \longrightarrow \displaystyle\sum_{f'=1}^{N_f} (U_L)_{ff'} Q^{f'}_L \\ Q^f_R \longrightarrow Q^f_R \end{cases} \qquad \mathrm{U}(N_f)_R : \begin{cases} Q^f_L \longrightarrow Q^f_L \\ Q^f_R \longrightarrow \displaystyle\sum_{f'=1}^{N_f} (U_R)_{ff'} Q^{f'}_R \end{cases}$$

Decomposing $\mathrm{U}(N) = \mathrm{SU}(N) \times \mathrm{U}(1)$, the global symmetry group can be written as

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_L \times \mathrm{U}(1)_R$$

Alternatively, we can write the global group in terms of its **vector** and **axial factors**

$$\mathrm{SU}(N_f)_V\times\mathrm{SU}(N_f)_A\times\mathrm{U}(1)_B\times\mathrm{U}(1)_A$$

where

$$\mathrm{SU}(N_f)_V:\left\{\begin{array}{l} Q_L^f\longrightarrow\displaystyle\sum_{f'=1}^{N_f}U_{ff'}Q_L^{f'} \\ Q_R^f\longrightarrow\displaystyle\sum_{f'=1}^{N_f}U_{ff'}Q_R^{f'}\end{array}\right.\qquad \mathrm{SU}(N_f)_A:\left\{\begin{array}{l} Q_L^f\longrightarrow\displaystyle\sum_{f'=1}^{N_f}U_{ff'}Q_L^{f'} \\ Q_R^f\longrightarrow\displaystyle\sum_{f'=1}^{N_f}U^{-1}_{ff'}Q_R^{f'}\end{array}\right.$$

and

$$\mathrm{U}(1)_B:\left\{\begin{array}{l} Q_L^f\longrightarrow e^{i\alpha}Q_L^f \\ Q_R^f\longrightarrow e^{i\alpha}Q_R^f\end{array}\right.\qquad \mathrm{U}(1)_A:\left\{\begin{array}{l} Q_L^f\longrightarrow e^{i\alpha}Q_L^f \\ Q_R^f\longrightarrow e^{-i\alpha}Q_R^f\end{array}\right.$$

The classically conserved currents associated with the axial factors are:

$$J^\mu_{\mathrm{A}}=\sum_{f=1}^{N_f}\overline{Q}^f\gamma^\mu\gamma_5Q^f\qquad J^{I\mu}_{\mathrm{A}}=\sum_{f,f'=1}^{N_f}\overline{Q}^f\gamma^\mu\gamma_5(T^I)_{ff'}Q^{f'}$$

However, at low energies **chiral symmetry is spontaneously broken** by fermion condensation

$$\langle\overline{Q}^f Q^f\rangle \neq 0$$

which leaves only the vector subgroups

$$\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R \longrightarrow \mathrm{SU}(N_f)_V \times \mathrm{U}(1)_B$$

Associated with the breaking of $\mathrm{SU}(N_f)_A$ we have $N_f^2 - 1$ Goldstone bosons

$$|\pi^I(p)\rangle \qquad\qquad I = 1, \ldots, N_f^2 - 1$$

satisfying

$$\langle\Omega|J_{\mathrm{A}}^{\mu I}(x)|\pi^J(p)\rangle \neq 0$$

Using translational invariance,

$$\langle\Omega|J_{\rm A}^{\mu I}(x)|\pi^J(p)\rangle = \langle\Omega|e^{iP\cdot x}J_{\rm A}^{\mu I}(0)e^{-iP\cdot x}|\pi^J(p)\rangle = e^{-ip\cdot x}\langle\Omega|J_{\rm A}^{\mu I}(0)|\pi^J(p)\rangle$$

Lorentz and group covariance implies now

$$\langle\Omega|J_{\rm A}^{\mu I}(x)|\pi^J(p)\rangle = if_\pi p^\mu \delta^{IJ}e^{-ip\cdot x}$$

with f_π is a constant.

Adding quark masses results in a nonvanishing mass for the (pseudo)Goldstone bosons

$$m_\pi^2 \ll m_N^2$$

Then we have

$$\langle\Omega|\partial_\mu J_{\rm A}^{\mu I}(x)|\pi^J(p)\rangle = f_\pi m_\pi^2 \delta^{IJ}e^{-ip\cdot x}$$

A short detour to the LSZ reduction formula

The basic observables in QFT are **scattering (S-matrix) amplitudes**…
How do we compute them in terms of correlation functions?

$$\langle\Omega|T[\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)]|\Omega\rangle \qquad {}_{\rm out}\langle\mathbf{p}_1',\mathbf{p}_2'|\mathbf{p}_1,\mathbf{p}_2\rangle_{\rm in}$$

?

The basic idea is to "reduce" on-shell particles using

$$\phi(x)_{\rm out}|\Omega\rangle_{\rm out} = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{ip\cdot x}}{2\omega_{\mathbf{p}}}|\mathbf{p}\rangle_{\rm out} \qquad \phi(x)_{\rm in}|\Omega\rangle_{\rm in} = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{ip\cdot x}}{2\omega_{\mathbf{p}}}|\mathbf{p}\rangle_{\rm in}$$

$$\langle\mathbf{p}|\mathbf{q}\rangle = (2\pi)^3 2\omega_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})$$

$${}_{\rm out}\langle\mathbf{p}|\phi(x)_{\rm out}|\Omega\rangle_{\rm out} = {}_{\rm in}\langle\mathbf{p}|\phi(x)_{\rm in}|\Omega\rangle_{\rm in} = e^{ip\cdot x}$$

Inside correlation functions, in- and out-fields are obtained asymptotically as

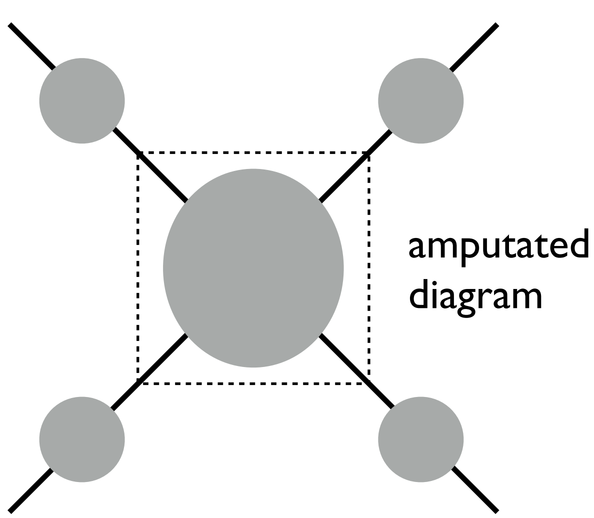
$$\lim_{x^0\to-\infty}\phi(x) = Z^{\frac{1}{2}}\phi(x)_{\rm in} \qquad \lim_{x^0\to\infty}\phi(x) = Z^{\frac{1}{2}}\phi(x)_{\rm out}$$

With these ingredients, one arrives at the **Lehmann-Symanzik-Zimmermann** reduction formula:

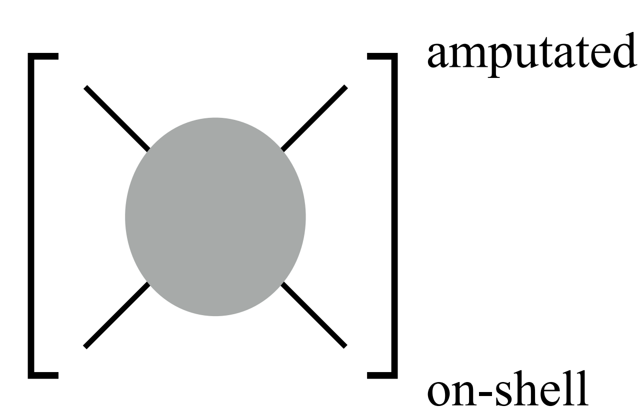
$$
\begin{aligned}
{}_{\text{out}}\langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2 \rangle_{\text{in}} &= \text{disconnected terms} \\
&+ (iZ^{-\frac{1}{2}})^4 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{-ip_1\cdot y_1 - ip_2\cdot y_2 + ip_1'\cdot x_1 + ip_2'\cdot x_2} \\
&\quad \times (\Box_{x_1} + m^2)(\Box_{x_2} + m^2)(\Box_{y_1} + m^2)(\Box_{y_2} + m^2)\langle \Omega | T[\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)|\Omega\rangle
\end{aligned}
$$

LSZ rule of thumb: $a(\mathbf{p})^\dagger \quad \longrightarrow \quad iZ^{-\frac{1}{2}} \int d^4x \, e^{-ip\cdot x}(\Box + m^2)\phi(x)$

The terms $iZ^{-\frac{1}{2}}(\Box + m^2)$ cancel the external propagators



$$
{}_{\text{out}}\langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2 \rangle_{\text{in}} =
$$



Here we used free canonical fields, but the **LSZ** formula **holds** for any **interpolating field** satisfying

$$\langle \mathbf{p}|\phi(x)|\Omega\rangle = e^{ip\cdot x}$$

For example, we could have used as well

$$\Phi(x) \equiv -\frac{1}{m^2}\Box\phi(x)$$

Since $p^2 = m^2$

$$\langle \mathbf{p}|\Phi(x)|\Omega\rangle = -\frac{1}{m^2}\Box\langle \mathbf{p}|\phi(x)|\Omega\rangle = \frac{p^2}{m^2}e^{ip\cdot x} = e^{ip\cdot x}$$

and the field $\Phi(x)$ is a good interpolating field and can be used to compute the S-matrix amplitudes (physics should not depend on local field redefinitions!)

back to pions…

$$\langle\Omega|\partial_\mu J_{\rm A}^{\mu I}(x)|\pi^J(p)\rangle = f_\pi m_\pi^2 \delta^{IJ} e^{-ip\cdot x}$$

We particularize the analysis to the u and d quarks $(N_f = 2)$.

Then, the divergence of the axial current is an **interpolating field** for the pions. In particular,

$$\phi_{\pi^0}(x) = \frac{1}{f_\pi m_\pi^2}\partial_\mu J_{\rm A}^{\mu 0} \qquad \longrightarrow \qquad \langle\Omega|\phi_{\pi^0}(x)|\pi^0(p)\rangle = e^{-ip\cdot x}$$

This field that can be used in the Lehmann-Symanzik-Zimmermann reduction formula to create pion asymptotic states, e.g. for $\pi^0 \longrightarrow 2\gamma$

$$\langle \mathbf{k}_1,\lambda_1;\mathbf{k}_2,\lambda_2|\pi^0(p)\rangle = \int d^4x\, e^{-ip\cdot x}(\Box + m_\pi^2)\langle \mathbf{k}_1,\lambda_1;\mathbf{k}_2;\lambda_2|\phi_{\pi^0}(x)|\Omega\rangle$$

$$= \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2}\int d^4x\, e^{-ip\cdot x}\langle \mathbf{k}_1,\lambda_1;\mathbf{k}_2,\lambda_2|\partial_\mu J_{\rm A}^{\mu 0}(x)|\Omega\rangle$$

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle = \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} \int d^4x \, e^{-ip\cdot x} \langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \partial_\mu J_{\rm A}^{\mu 0}(x) | \Omega \rangle$$

and applying again the LSZ formula for the photons, the result is

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle = \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} \int d^4x \int d^4y \int d^4z \, e^{-ip\cdot x + ik_1\cdot y + ik_2 \cdot z}$$

$$\times \langle \Omega | T[\partial_\mu J_{\rm A}^{\mu 0}(x) J_{\rm em}^{\alpha}(y) J_{\rm em}^{\beta}(z)] | \Omega \rangle \, \epsilon_\alpha(\mathbf{k}_1) \epsilon_\beta(\mathbf{k}_2)$$

In momentum space

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle = \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2) p^\mu \Gamma_{\mu\alpha\beta}(k_1, k_2) \epsilon^\alpha(\mathbf{k}_1) \epsilon^\beta(\mathbf{k}_2)$$

where from **Lorentz invariance** we know that (see above our discussion on the IR interpretation of the anomaly)

$$i\Gamma_{\mu\alpha\beta}(k_1, k_2) = f(p^2) p_\mu \epsilon_{\alpha\beta\sigma\lambda} k_1^\sigma k_2^\lambda$$

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p)\rangle = \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} \int d^4x\, e^{-ip\cdot x} \langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \partial_\mu J_{\rm A}^{\mu 0}(x) | \Omega\rangle$$

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In momentum space

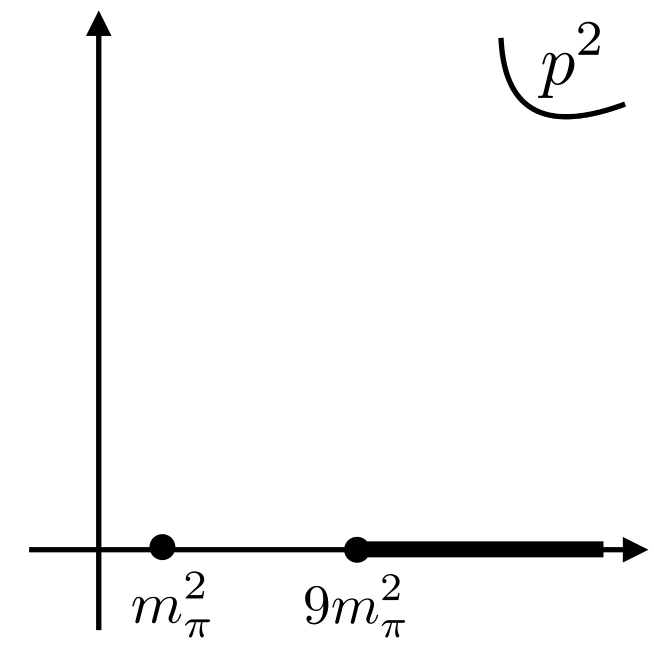
remember: $\langle 0 | J_{\rm A}^{\mu}(0) | p, q\rangle_{\mathscr{A}} = \Gamma^{\mu\alpha\beta}(p,q) \widetilde{\mathscr{A}_\alpha}(p) \widetilde{\mathscr{A}_\beta}(q) \Big|_{p^2=q^2=0}$

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p)\rangle = \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2) p^\mu \Gamma_{\mu\alpha\beta}(k_1, k_2) \epsilon^\alpha(\mathbf{k}_1)\epsilon^\beta(\mathbf{k}_2)$$

where from **Lorentz invariance** we know that (see above our discussion on the IR interpretation of the anomaly)

$$i\Gamma_{\mu\alpha\beta}(k_1, k_2) = f(p^2) p_\mu \epsilon_{\alpha\beta\sigma\lambda} k_1^\sigma k_2^\lambda$$

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle \sim \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} p^2 f(p^2)$$



Naively, one could argue that $f(p^2)$ has to **behave smoothly** at low momenta, its only singularities being the one-particle pole and thresholds at higher energies (**PCAC hypothesis**).

This automatically implies that the amplitude vanishes as $p^2 \longrightarrow 0$

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle \bigg|_{p^2 = m_\pi^2} \approx \langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle \bigg|_{p^2 = 0} = 0$$

The decay $\pi^0 \longrightarrow 2\gamma$ is **suppressed! (Sutherland-Veltman theorem**)

However $\Gamma(\pi^0 \to 2\gamma)_{\rm exp} = 7.82 \pm 0.14({\rm stat.}) \pm 0.17({\rm syst.})\,{\rm eV}$

(PriMex collaboration, 2011)

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle \sim \frac{m_\pi^2 - p^2}{f_\pi m_\pi^2} p^2 f(p^2)$$

The **key ingredient** to clarify the Sutherland-Veltman paradox is the existence of the **anomaly**!!

As we saw, $f(p^2)$ is not smooth, but it has a pole at $p^2 = 0$

$$f(p^2) = \frac{i}{2\pi} \frac{c}{p^2}$$

where c is the **group theory factor** multiplying the triangle diagram, that depends on the number of fermions running in the triangle diagram. Then

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle = \frac{c}{2\pi^2 f_\pi} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2)\, \epsilon_{\mu\nu\alpha\beta} k_1^\mu k_2^\nu \epsilon_\alpha(\mathbf{k}_1) \epsilon_\beta(\mathbf{k}_2)$$

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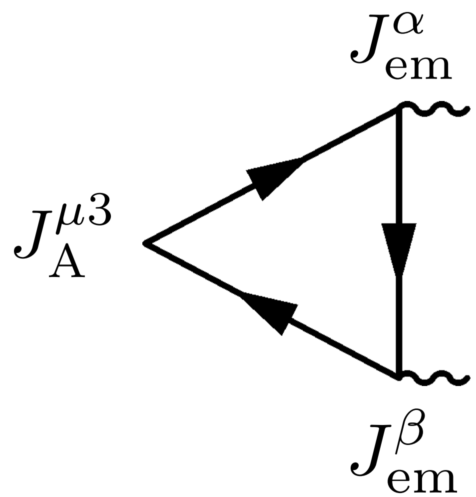
The constant c can be computed as the group theory factor multiplying the diagram



$$: \quad c = \sum_{f=1}^{N_f} q_f^2 T^3_{ff} = \sum_{f=1}^{N_f} q_f^2 \frac{\sigma^3_{ff}}{2} \qquad \left(T^a_{ij} = \frac{\sigma^a_{ij}}{2} \right)$$

Hence,

$$c = N_c \left[\left(\frac{2e}{3} \right)^2 \frac{1}{2} - \left(-\frac{e}{3} \right)^2 \frac{1}{2} \right] = \frac{e^2 N_c}{6}$$

and then

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle = \frac{e^2 N_c}{12\pi^2 f_\pi} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2)\, \epsilon_{\mu\nu\alpha\beta} k_1^\mu k_2^\nu \epsilon_\alpha(\mathbf{k}_1) \epsilon_\beta(\mathbf{k}_2)$$

$$\langle \mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2 | \pi^0(p) \rangle = \frac{c}{2\pi^2 f_\pi}(2\pi)^4 \delta^{(4)}(p - k_1 - k_2)\, \epsilon_{\mu\nu\alpha\beta} k_1^\mu k_2^\nu \epsilon_\alpha(\mathbf{k}_1) \epsilon_\beta(\mathbf{k}_2)$$

We compute now the width for the unpolarized decay as

$$\Gamma(\pi^0 \to 2\gamma) = \frac{e^4 N_c^2}{288\pi^2 m_\pi f_\pi} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{1}{2|\mathbf{k}_1|} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{1}{2|\mathbf{k}_2|} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2)$$

$$\times \sum_{\text{polarizations}} \left| \epsilon_{\mu\nu\alpha\beta} k_1^\mu k_2^\nu \epsilon_\alpha(\mathbf{k}_1) \epsilon_\beta(\mathbf{k}_2) \right|^2$$

which gives

$$\Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2 N_c^2 m_\pi^3}{576\pi^3 f_\pi^2} \quad \longrightarrow \quad \Gamma(\pi^0 \to 2\gamma) = 7.73\,\text{eV}$$

which is pretty close to the experimental value

$$\Gamma(\pi^0 \to 2\gamma)_{\text{exp}} = 7.82 \pm 0.14(\text{stat.}) \pm 0.17(\text{syst.})\,\text{eV}$$

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color-counting observable!

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Aside: how do we compute these observables?

- **Pion decay constant:**

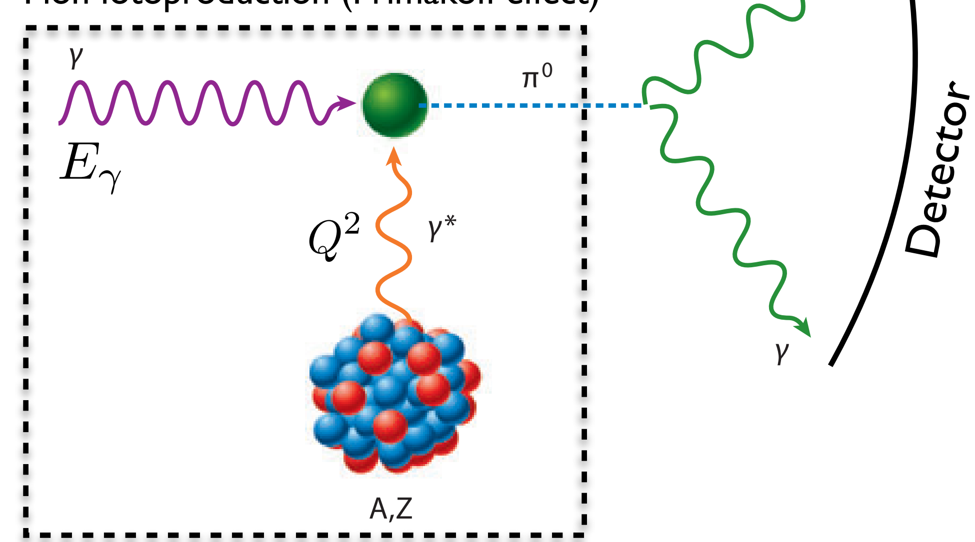
$$\pi^+ \longrightarrow \mu^+\nu_\mu \quad \Longrightarrow \quad \Gamma(\pi^+ \to \mu^+\nu_\mu) = \frac{1}{4\pi} G_F^2 \cos^2\theta_C f_\pi^2 m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

A measurement of the charged muon lifetime leads to a value of the pion decay constant.

- **Neutral pion decay width:**

The quoted values for the pion width were obtained using the Primakoff effect by the PriMex collaboration.

$$\gamma A \longrightarrow \pi^0 A$$



From Ann. Rev. Nucl. Part. Sci. 61 (2011) 1

The production cross section is

$$\frac{d\sigma_P}{d\Omega} = \Gamma(\pi^0 \to 2\gamma)\frac{8\alpha Z^2}{m_\pi^3}\frac{\beta^3 E_\gamma^4}{Q^4}|F_{\rm em}(Q^2)|^2 \sin^2\theta_\pi$$

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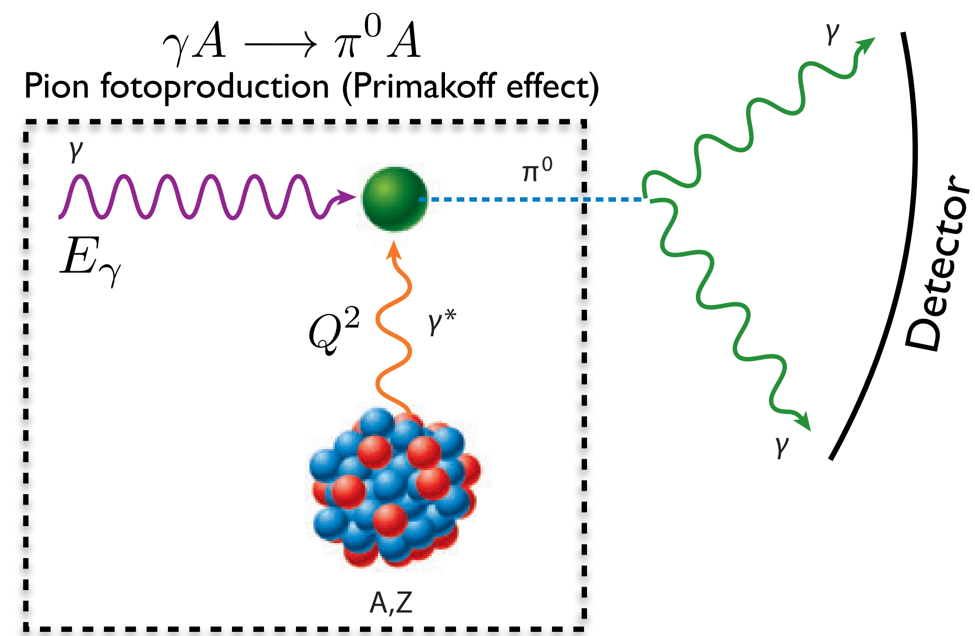
Lab velocity and angle of the pion

Pion fotoproduction is "dual" to pion decay

Nucleus electromagnetic form factor

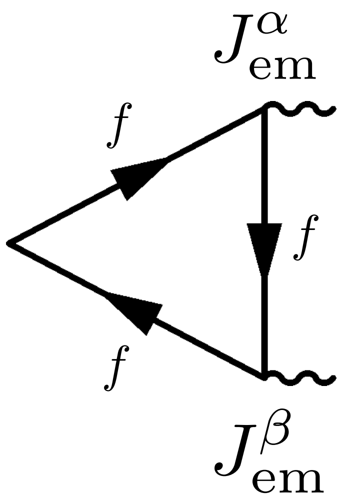


From Ann. Rev. Nucl. Part. Sci. 61 (2011) 1

This is how the axial anomaly **explains** the electromagnetic decay of the pion!

What about the other isospin components of $J_{\mathrm{A}}^{\mu I}(x)$?

$\sum_f J_{\mathrm{A}}^{\mu I}$

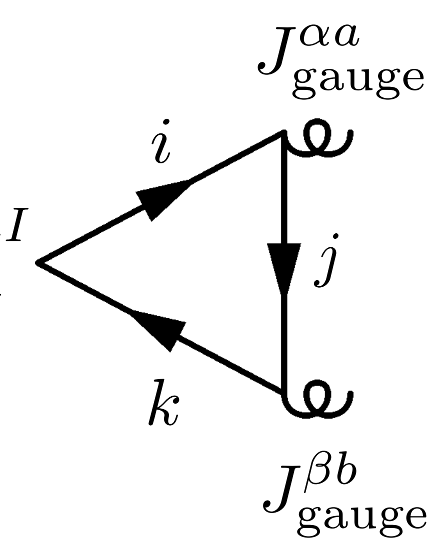


$$: \quad \partial_\mu \langle J_{\mathrm{A}}^{\mu I}(x)\rangle_{\mathscr{A}} = \frac{1}{16\pi^2}\left[\sum_{f=1}^{N_f} q_f^2 \frac{\sigma^I_{ff}}{2}\right]\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

But $\sigma^2_{ff} = \sigma^3_{ff} = 0$ (no sum!) so the triangle vanishes for $I = 1, 2$.

Besides, QCD does not contribute to the anomaly of any component:

$\sum_{i,j,k} J_{\mathrm{A}}^{\mu I}$



$$\sim \mathrm{Tr}\left[\frac{\sigma^I}{2}\{\tau^a,\tau^b\}\right] = \mathrm{Tr}\left(\frac{\sigma^I}{2}\right)\mathrm{Tr}\left[\{\tau^a,\tau^b\}\right] = 0$$

$$\mathrm{SU}(2)_V \times \mathrm{SU}(2)_A \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A$$

$$\mathrm{SU}(2)_V \times \mathrm{U}(1)_B$$

We have three pions as (pseudo)Goldstone bosons of the spontaneous breaking of $\mathrm{SU}(2)_A$.

What about the (pseudo)Goldstone boson associated with $\mathrm{U}(1)_A$?

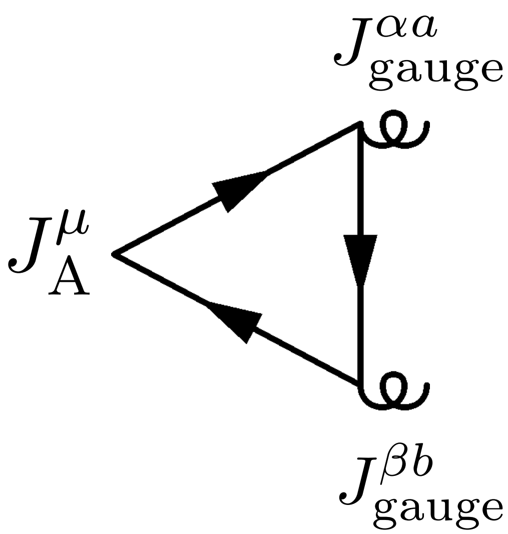
The η-meson seems to have the right quantum numbers… but

$$m_\eta \approx 547\,\mathrm{MeV} \gg m_\pi \approx 135\,\mathrm{MeV}$$

It is far too heavy! In fact, Weinberg has showed that

$m_0 \leq \sqrt{3} m_\pi$ → nowhere to be found!

Maybe $\mathrm{U}(1)_A$ is anomalous in the first place…



$$: \quad \partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{N_f}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)\right] \neq 0$$

This is not however the whole story, since the anomaly is a total derivative

$$\frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\right) = \frac{1}{4\pi^2}\partial_\mu \mathrm{Tr}\left[\epsilon^{\mu\nu\alpha\beta}\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta + \frac{2}{3}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right)\right]$$

$$\equiv 2\partial_\mu \mathscr{K}^\mu$$

so we write

$$\partial_\mu \langle J^\mu_{\rm A}\rangle_{\mathscr{A}} = 2N_f\partial_\mu\mathscr{K}^\mu \quad \longrightarrow \quad \partial_\mu\Big(\langle J^\mu_{\rm A}\rangle_{\mathscr{A}} - 2N_f\mathscr{K}^\mu\Big) = 0$$

$$\partial_\mu\Big(\langle J^\mu_{\rm A}\rangle_{\mathscr{A}} - 2N_f\mathscr{K}^\mu\Big) = 0$$

We can define then a new "improved" conserved current

$$\mathscr{J}^\mu \equiv \langle J^\mu_{\rm A}\rangle_{\mathscr{A}} - 2N_f\mathscr{K}^\mu$$

which, however, is not gauge invariant. Still, we can compute the conserved charge

$$Q_5' = \int d^3\mathbf{r}\,\Big(\langle J^0_{\rm A}\rangle_{\mathscr{A}} - 2N_f\mathscr{K}^0\Big)$$

$$= Q_5 - \frac{N_f}{4\pi^2}\int d^3\mathbf{r}\,\epsilon^{ijk}\mathrm{Tr}\,\left(\mathscr{A}_i\partial_j\mathscr{A}_k + \frac{2}{3}\mathscr{A}_i\mathscr{A}_j\mathscr{A}_k\right)$$

Chern-Simons action!

$$S_{\rm CS} = \frac{1}{8\pi^2}\int d^3\mathbf{r}\,\epsilon^{ijk}\mathrm{Tr}\,\left(\mathscr{A}_i\partial_j\mathscr{A}_k + \frac{2}{3}\mathscr{A}_i\mathscr{A}_j\mathscr{A}_k\right)$$

$$Q_5' = Q_5 - \frac{N_f}{4\pi^2}\int d^3\mathbf{r}\,\epsilon^{ijk}\mathrm{Tr}\left(\mathscr{A}_i\partial_j\mathscr{A}_k + \frac{2}{3}\mathscr{A}_i\mathscr{A}_j\mathscr{A}_k\right)$$

But the Chern-Simons term, is not invariant under "large" gauge transformations with **winding number** n,

$$S_{\mathrm{CS}} \longrightarrow S_{\mathrm{CS}} + n$$

Thus, the charge transforms as

$$\mathscr{G}_n Q_5' \mathscr{G}_n^{-1} = Q_5' - 2nN_f$$

This implies that chiral transformations are broken by the θ-vacuum

$$e^{i\varphi Q_5'}|\theta\rangle = |\theta + 2N_f\varphi\rangle$$

However, this spontaneous breaking of the axial U(1) does not produce Goldstone bosons ('t Hooft's solution of the U(1) problem, we'll come back to it at the end of the course…).

Phenomenology of anomalies II: Anomaly cancellation

We have seen that the condition for the cancellation of anomalies reads:

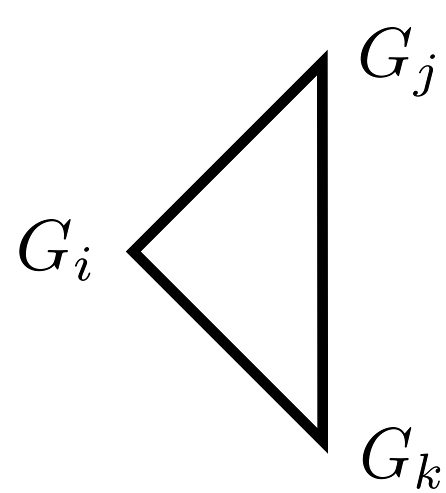
$$\sum_{\text{right-handed}} \text{Tr}\left[T_{+}^{a}\{T_{+}^{b}, T_{+}^{c}\}\right] - \sum_{\text{left-handed}} \text{Tr}\left[T_{-}^{a}\{T_{-}^{b}, T_{-}^{c}\}\right] = 0$$

This provides a nontrivial constraint on the particle spectrum and group representations of a theory and it is a useful tool in **model building**

Here we analyze two cases:

- The standard model
- Minimal supersymmetric standard model

Remember that if the gauge group is a **product** group $G_1 \times \ldots \times G_n$ we have to consider "mixed" triangles as well



$$i, j, k = 1, \ldots, n$$

The standard model

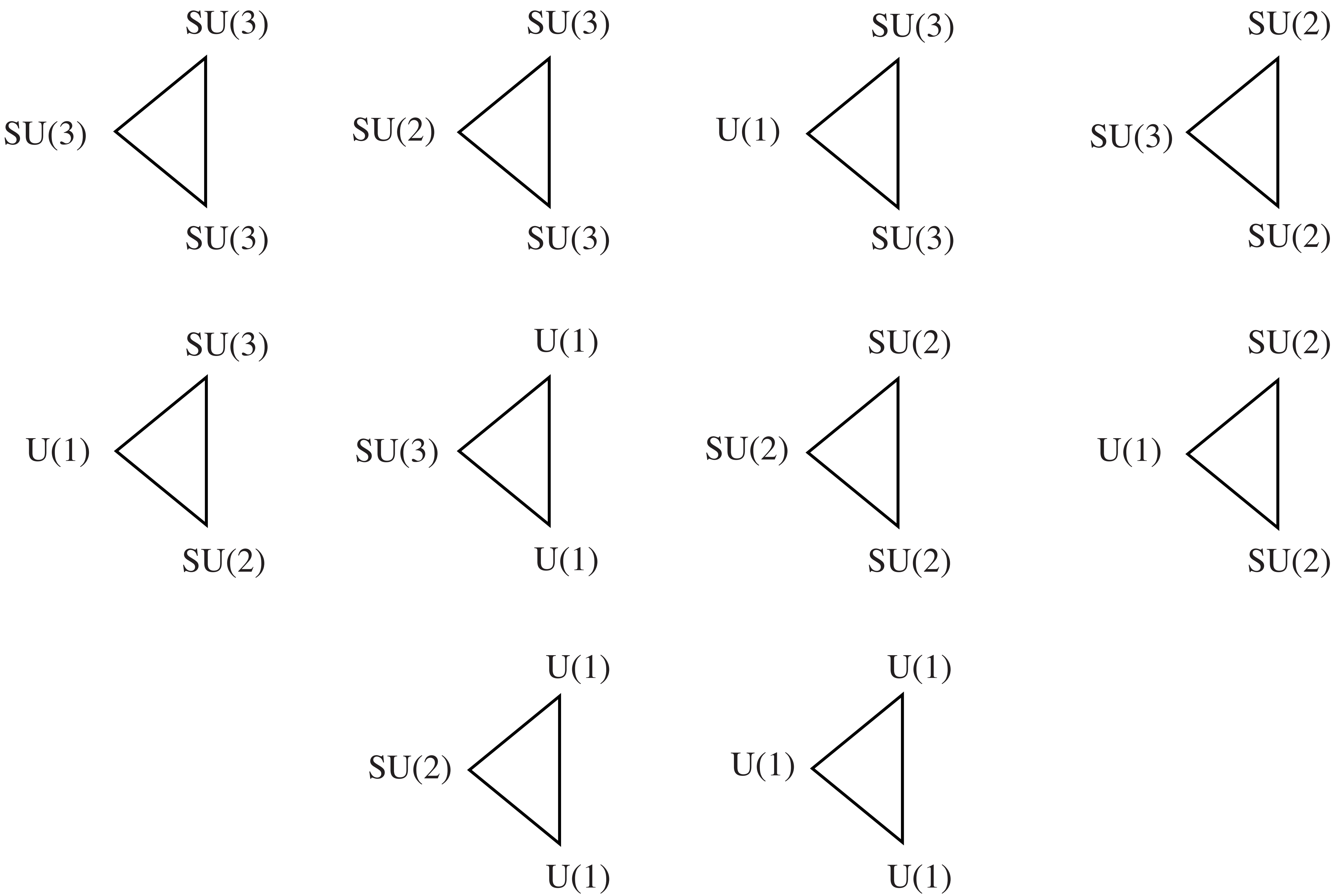
For the purpose of anomaly cancellation, we only have to care about chiral fermions (i.e. leptons and quarks). Denoting their representations of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y$ by

$$(n_c, n_W)_Y$$

the **fermion content** of the theory is

	quarks		leptons	
Left-handed fermions:	$(3,2)^L_{\frac{1}{6}}$		$(1,2)^L_{-\frac{1}{2}}$	x 3 families
Right-handed fermions:	$(3,1)^R_{\frac{2}{3}}$	$(3,1)^R_{-\frac{1}{3}}$	$(1,1)^R_{-1}$	

Symbolically, the ten anomaly coefficients to compute are:



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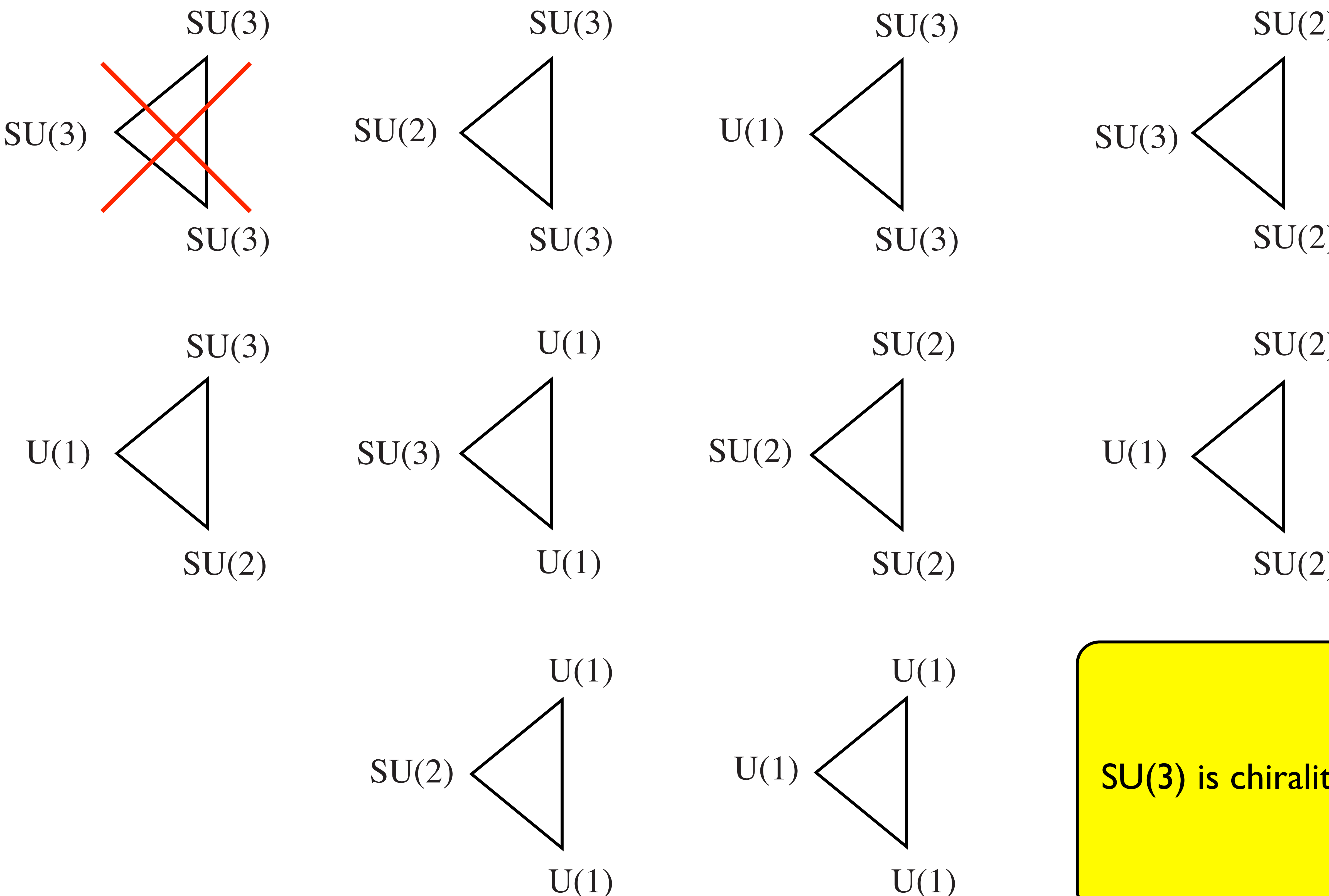

SU(3) is chirality-blind

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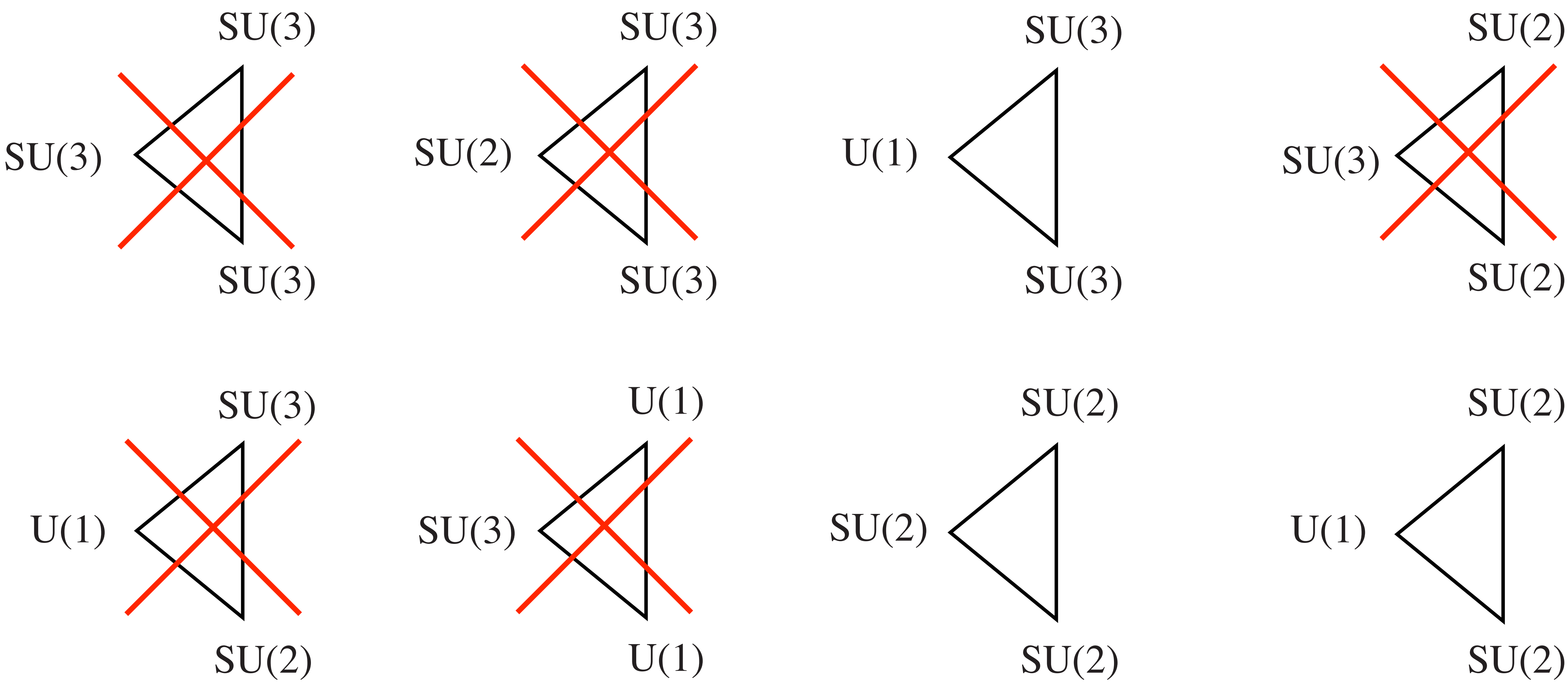

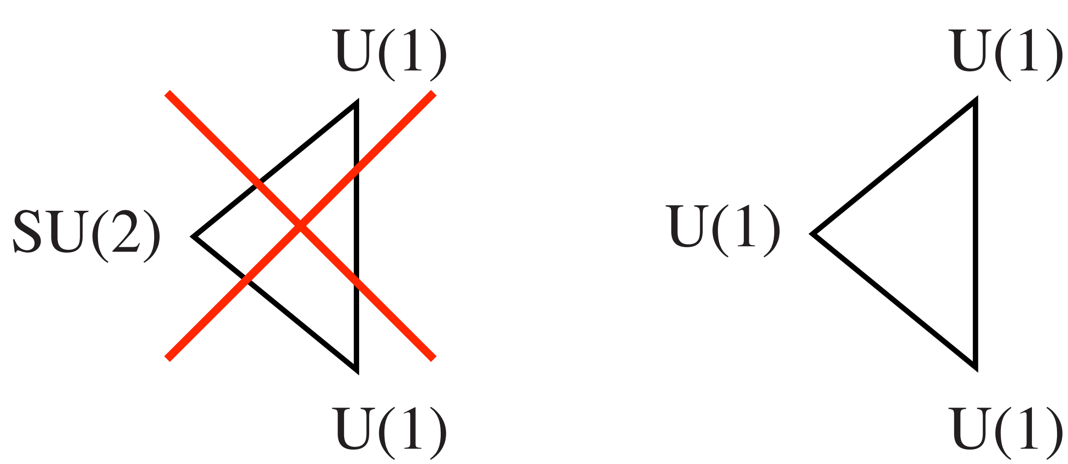

The generator of SU(2) and SU(3) are traceless

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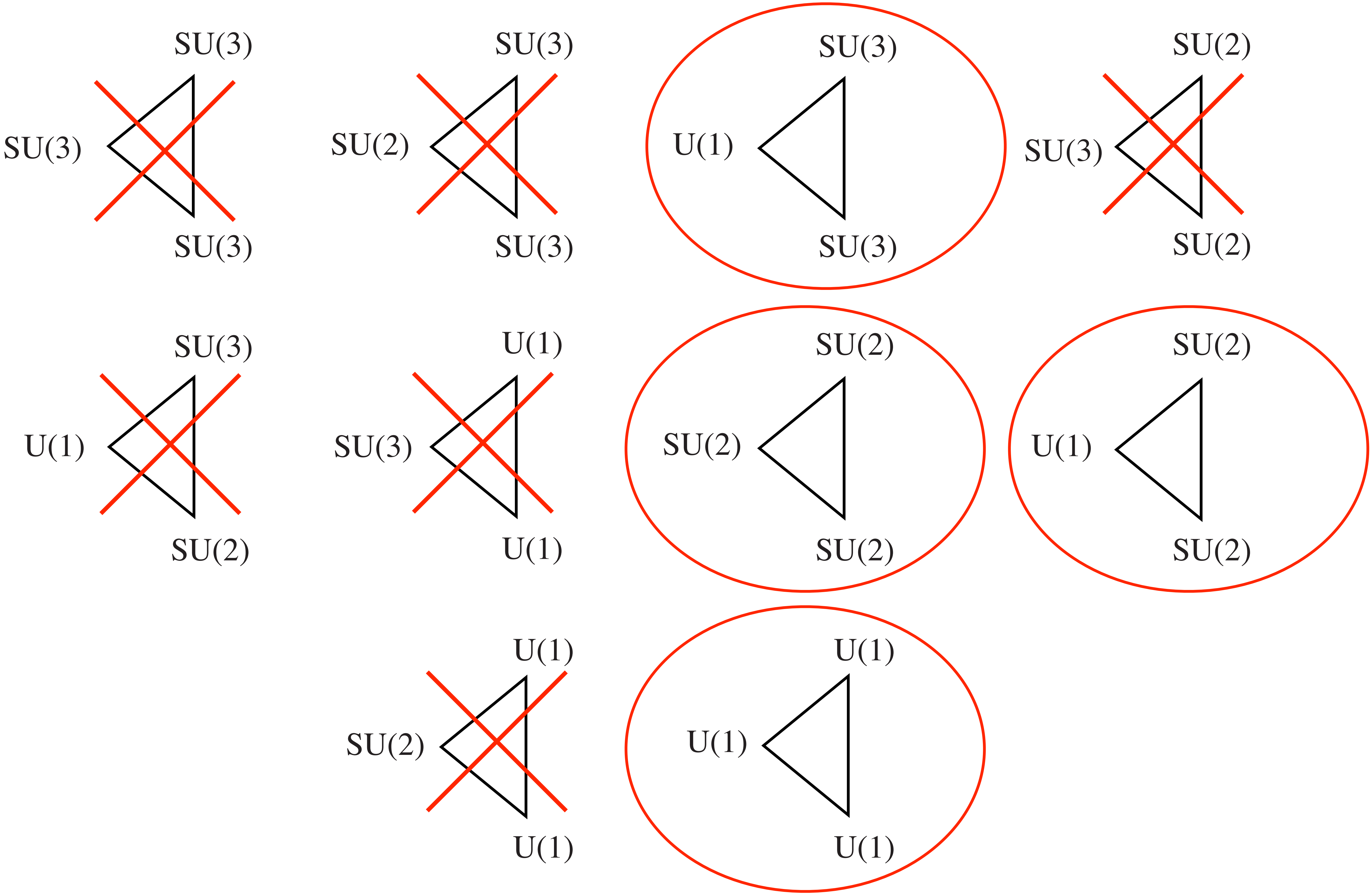
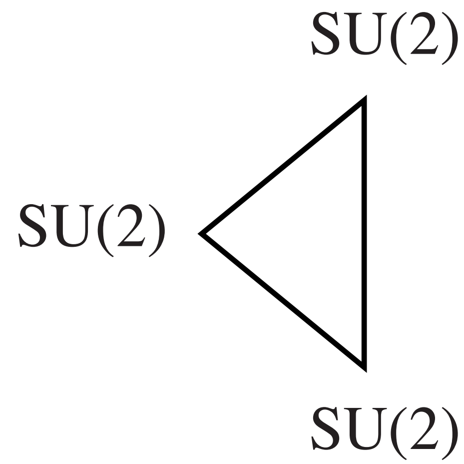
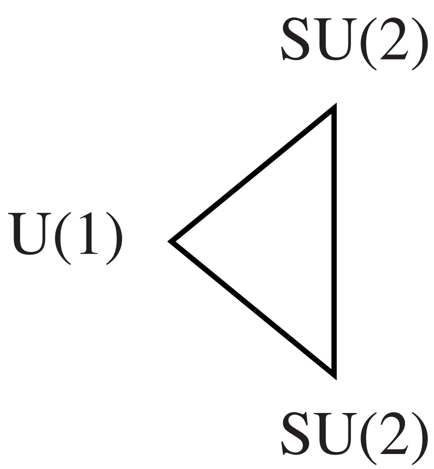




$$\sim \mathrm{Tr}\left[\sigma_i\{\sigma_j,\sigma_k\}\right] = 2(\mathrm{Tr}\,\sigma_i)\delta_{jk} = 0$$

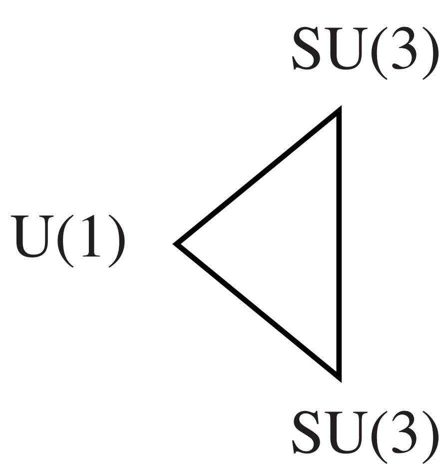
[SU(2) is a safe group]

$$\{\sigma_j,\sigma_k\} = 2\delta_{jk}$$



$$\sim \sum_L Y_L = \underbrace{3\times 2\times\left(\frac{1}{6}\right)}_{\text{quarks}} + \underbrace{2\times\left(-\frac{1}{2}\right)}_{\text{leptons}} = 0$$

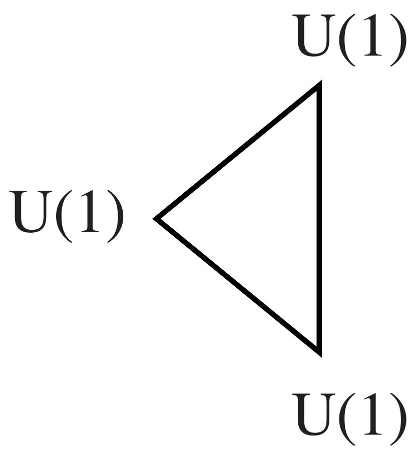
[right-handed fermions are SU(2) singlets]



$$\sim \sum_{q_L} Y_L - \sum_{q_R} Y_R = 3\times 2\times\left(\frac{1}{6}\right) - 3\times\left(\frac{2}{3}\right) - 3\times\left(-\frac{1}{3}\right) = 0$$

[leptons do not couple to gluons]

The strongest condition comes from the $\mathrm{U}(1)^3$ triangle:

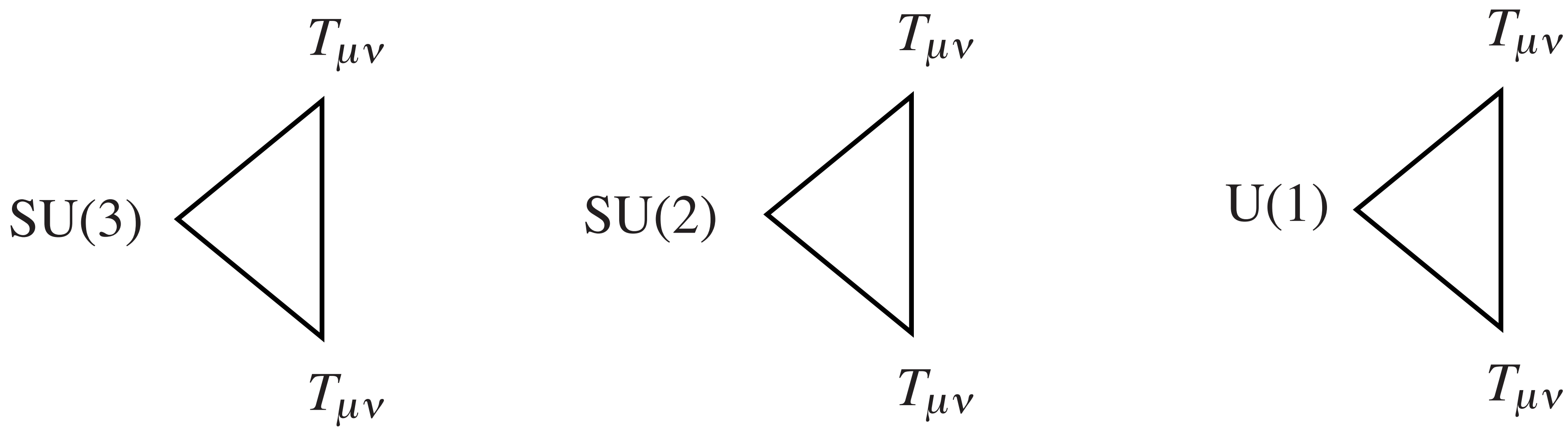


$$\sim \sum_L Y_L^3 - \sum_R Y_R^3 = 3\times 2\times\left(\frac{1}{6}\right)^3 + 2\times\left(-\frac{1}{2}\right)^3 - 3\times\left(\frac{2}{3}\right)^3$$

$$-3\times\left(-\frac{1}{3}\right)^3 - (-1)^3 = \underbrace{\left(-\frac{3}{4}\right)}_{\text{quarks}} + \underbrace{\left(\frac{3}{4}\right)}_{\text{leptons}} = 0$$

Hence, all **pure gauge anomalies** cancel in the standard model within each family!

Finally, we deal with the **mixed gauge-gravitational anomalies**:



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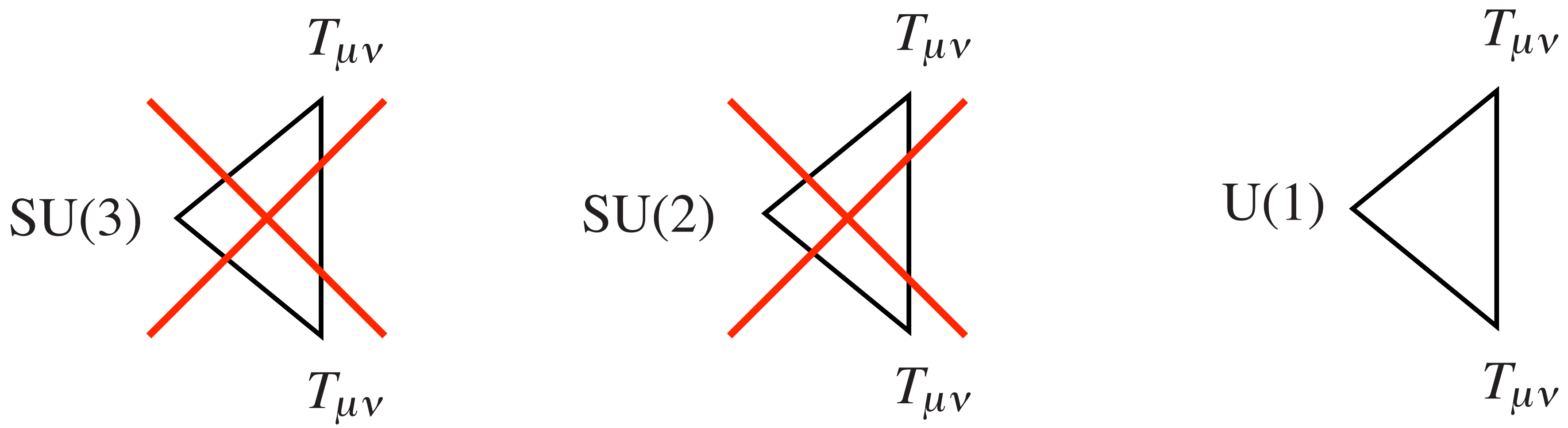

The generator of SU(2) and SU(3) are traceless

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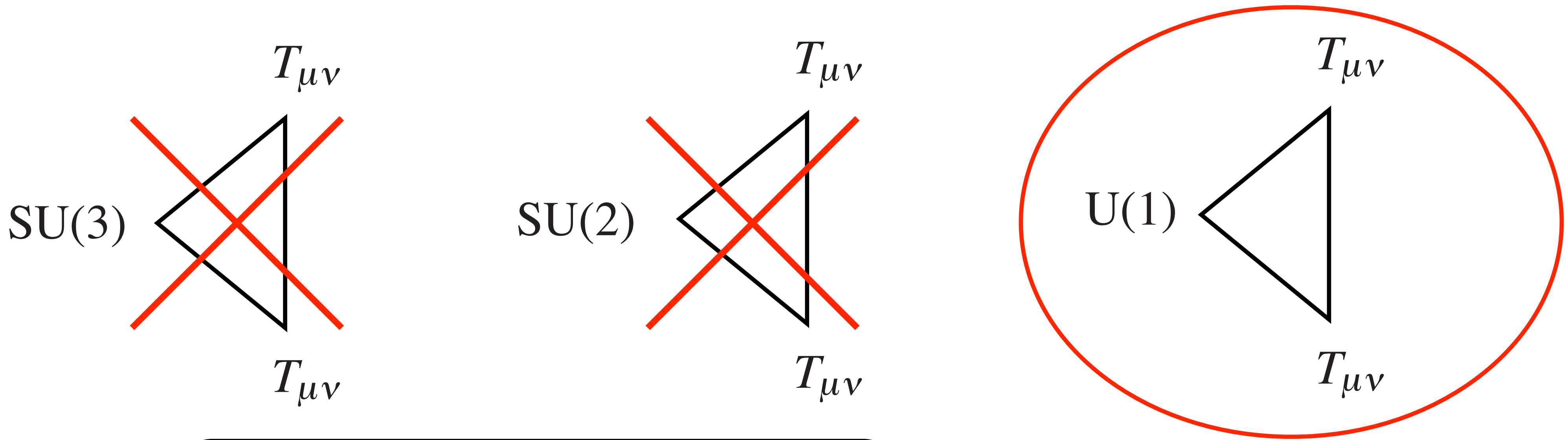
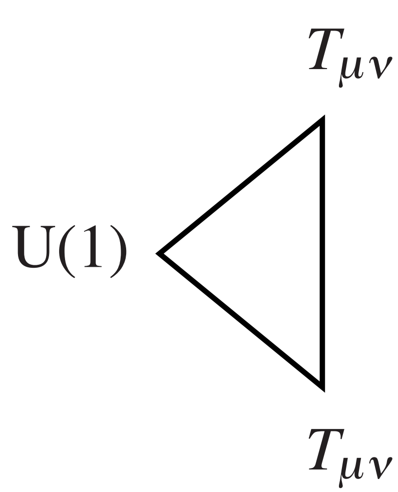


The generator of SU(2) and SU(3) are traceless

and for the last diagram we have



$$\sim \sum_L Y_L - \sum_R Y_R = 3 \times 2 \times \left(\frac{1}{6}\right) + 2 \times \left(-\frac{1}{2}\right) - 3 \times \left(\frac{2}{3}\right) - 3 \times \left(-\frac{1}{3}\right) - (-1) = 0$$

Thus, all **pure** and **mixed** gauge anomalies **cancel** and the standard model is **anomaly free**!

This cancellation is very **delicate** and severely constraints any extension of the standard model.

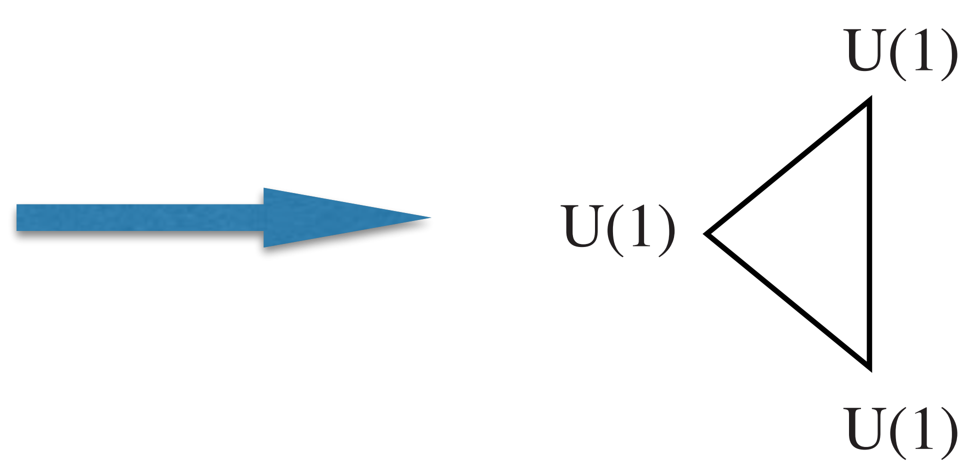
For example, the addition of a **sterile right-handed neutrino** is **innocuous**, since it does not contribute to the triangle:

right-handed neutrino: $(1,1)_0^R$

Since anomalies cancel within each family, we can also add any number of extra families.

However, one extra lepton (or quark) makes the theory inconsistent, e.g.

$(1,2)^L_{-\frac{1}{2}}$



$\neq 0$

Minimally supersymmetric standard model (MSSM)

In the minimal supersymmetric extension of the standard model, the spectrum is doubled

gauge bosons ($s = 1$)	⟷	gauginos ($s = \frac{1}{2}$)
quarks ($s = \frac{1}{2}$)	⟷	squarks ($s = 0$)
leptons ($s = \frac{1}{2}$)	⟷	sleptons ($s = 0$)
Higgs ($s = 0$)	⟷	Higgsino ($s = \frac{1}{2}$)

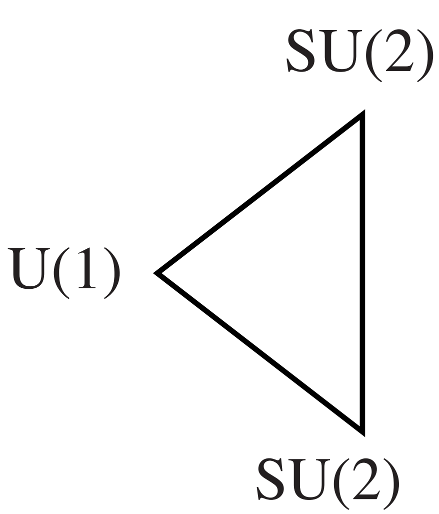
Since we knwo that all anomalies cancel in the standard model, we only have to worry about the new chiral fermions:

gauginos:
- gluino $(8,1)_0$
- wino $(1,3)_0$
- bino $(1,1)_0$

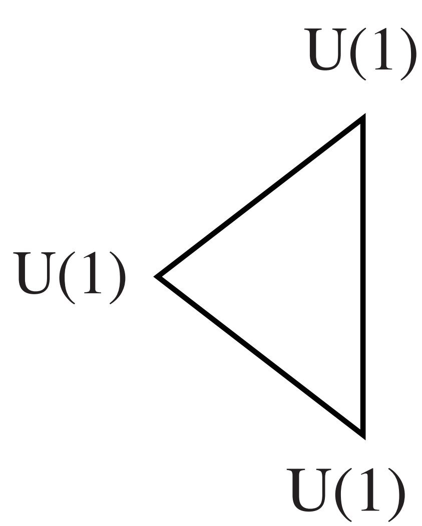
gluino, wino → The adjoint representation is **real** there are no anomalies!

bino → **singlet**

Higgsino: $(1,2)_{\frac{1}{2}}$



$\sim \dfrac{1}{2}$

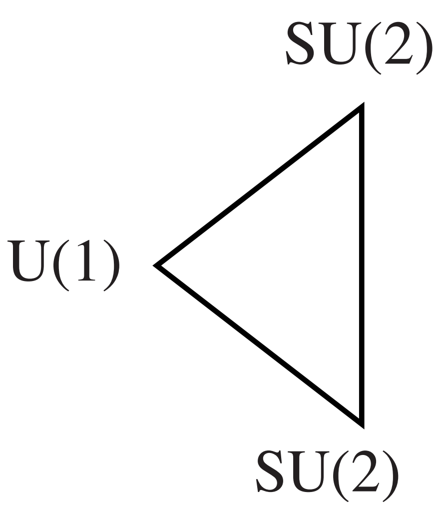


$\sim \left(\dfrac{1}{2}\right)^3$ → The MSSM with a single Higgsino is **anomalous**!

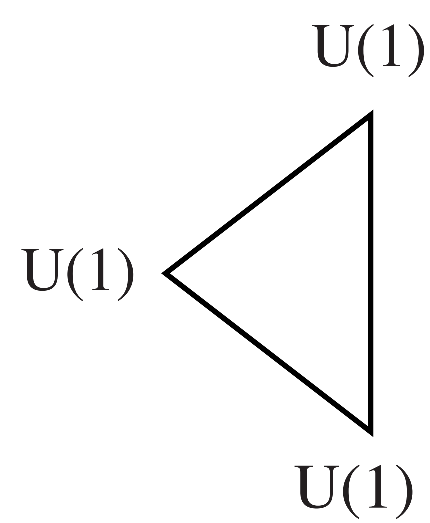
Thus, a ***second Higgs doublet*** with the same helicity and opposite hypercharge is required.

$$H_1 : \ (1,2)_{\frac{1}{2}} \qquad\qquad H_2 : \ (1,2)_{-\frac{1}{2}}$$

Now **all anomalies cancel** (a the second Higgs scalar doublet does not contribute to the SM anomaly!)

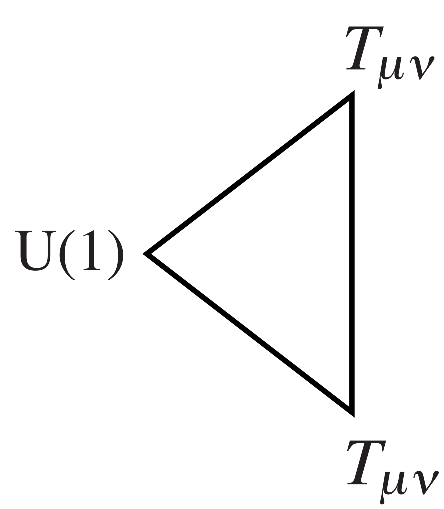


$$\sim \frac{1}{2} + \left(-\frac{1}{2}\right) = 0$$



$$\sim \left(\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^3 = 0$$

as well as the **mixed gauge-gravitational anomaly**



$$\sim \frac{1}{2} + \left(-\frac{1}{2}\right) = 0$$

Thus, the MSSM with two Higgs doublets is anomaly free.

The second Higgsino also cancels **Witten's global anomaly** (a theory with an odd number of SU(2) doublets is anomalous under "large" gauge transformations)

More on this later…

Phenomenology of anomalies III: Anomaly matching

Anomalies pose nontrivial constraints on theories confining below certain scale (e.g. QCD, technicolor, etc.)

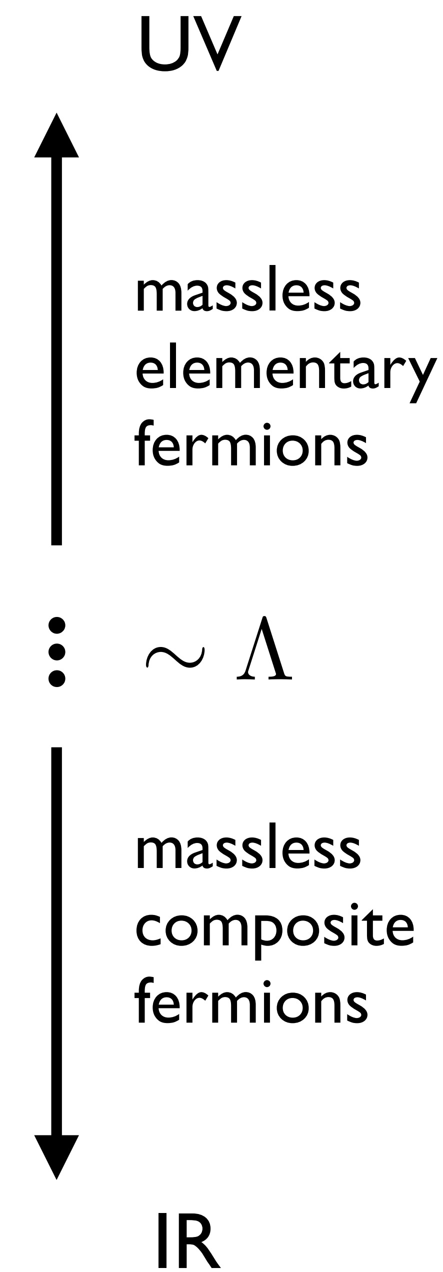
Let us assume the theory has a **global symmetry** group G such that:

The **UV degrees of freedom** transform in the representation

$$T^a \qquad\qquad (a=1,\ldots,\dim G)$$

whereas in the **IR** the massless composite fermions transform in

$$\widetilde{T}^a \qquad\qquad (a=1,\ldots,\dim G)$$



Next we **gauge** this global symmetry by coupling its Noether current to a nonabelian gauge field $B_\mu^a(x)$

$$\Delta S = g' \int d^4x\, B_\mu^a(x) J_G^{\mu a}(x)$$

But we have to be careful… if

$$\sum_{\text{right}-\text{handed}} \text{Tr}\left[T_+^a\{T_+^b, T_+^c\}\right] - \sum_{\text{left}-\text{handed}} \text{Tr}\left[T_-^a\{T_-^b, T_-^c\}\right] \neq 0$$

the theory has a gauge anomaly and it is no good.

This we solve by adding a number of **spectator fermions** coupling only to the gauge field $B_\mu^a(x)$ in such a way that they cancel the gauge anomaly

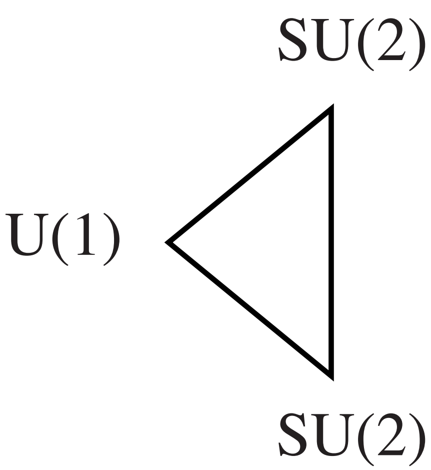
Since g' can be **tuned** to be as small as needed, the new fermions do not modify the original dynamics. (eventually we will take g' to zero)

An familiar **example** of spectator fermions.

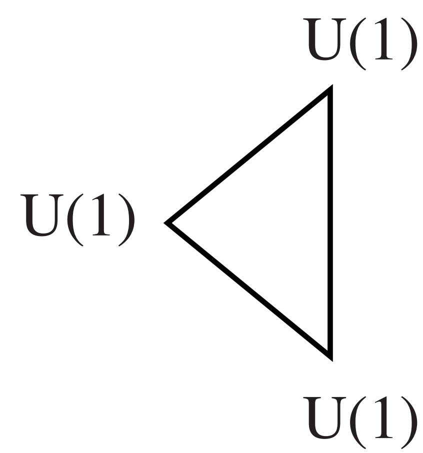
Let us consider **QCD**. The action of the $\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ electroweak group on the quarks can be regarded as a **global symmetry**

This global symmetry can be **gauged** it by coupling quarks to the electroweak gauge bosons.

But the resulting theory is anomalous



$$\sim \sum_{q_L} Y_{q_L} = 3\times 2\times\left(\frac{1}{6}\right) = 1$$



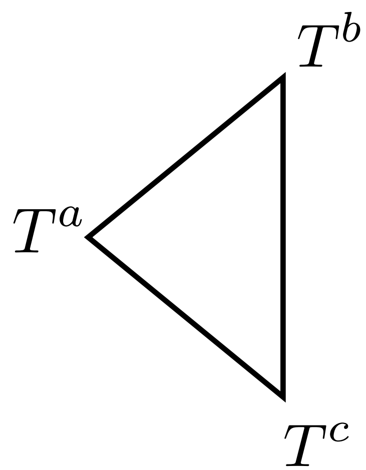
$$\sim \sum_{q_L} Y_{q_L}^3 - \sum_{q_R} Y_{q_R}^3 = -\frac{3}{4}$$

To cancel the anomaly **add the leptons**!

The new fermions do not interfere with the strong IR dynamics of QCD.

Now, anomalies are absent both in the UV and IR sectors:

In the **UV**, the anomaly cancellation condition simply reads



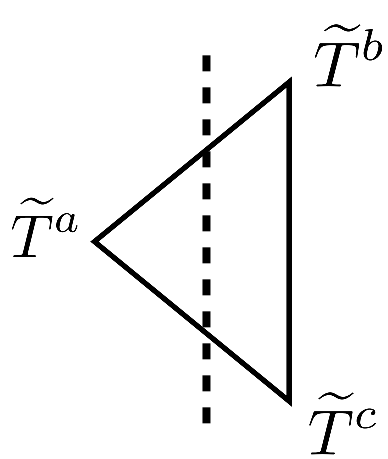
$$\sim \sum_L \mathrm{Tr}\left[T_L^a\{T_L^b, T_L^c\}\right] - \sum_R \mathrm{Tr}\left[T_R^a\{T_R^b, T_R^c\}\right] + \left(\begin{array}{c}\text{spectator}\\ \text{fermions}\end{array}\right) = 0$$

In the **IR** theory, it is more convenient to look at the IR picture of the anomaly

$$\mathrm{Im}\,\mathcal{A} \sim c_{\mathrm{IR}}\delta(q^2)$$

If the global group G **remains unbroken** at low energies, the IR contribution to the anomaly comes from "massless triangles"

$$\mathrm{Im}\,\mathcal{A} =$$



$$c_{\mathrm{IR}} = \sum_L \mathrm{Tr}\left[\widetilde{T}_L^a\{\widetilde{T}_L^b, \widetilde{T}_L^c\}\right] - \sum_R \mathrm{Tr}\left[\widetilde{T}_R^a\{\widetilde{T}_R^b, \widetilde{T}_R^c\}\right] + \left(\begin{array}{c}\text{spectator}\\ \text{fermions}\end{array}\right)$$

$$= 0$$

$$\sum_L \mathrm{Tr}\left[T_L^a\{T_L^b,T_L^c\}\right]-\sum_R \mathrm{Tr}\left[T_R^a\{T_R^b,T_R^c\}\right]+\left(\begin{array}{c}\text{spectator}\\ \text{fermions}\end{array}\right)=0$$

$$\sum_L \mathrm{Tr}\left[\widetilde{T}_L^a\{\widetilde{T}_L^b,\widetilde{T}_L^c\}\right]-\sum_R \mathrm{Tr}\left[\widetilde{T}_R^a\{\widetilde{T}_R^b,\widetilde{T}_R^c\}\right]+\left(\begin{array}{c}\text{spectator}\\ \text{fermions}\end{array}\right)=0$$

Gerard 't Hooft
(b. 1946)

Since the spectator fermions are weakly coupled, their contribution is the same in the UV and IR. This leads to **'t Hooft anomaly matching condition**:

$$\sum_L \mathrm{Tr}\left[T_L^a\{T_L^b,T_L^c\}\right]-\sum_R \mathrm{Tr}\left[T_R^a\{T_R^b,T_R^c\}\right]$$
$$=\sum_L \mathrm{Tr}\left[\widetilde{T}_L^a\{\widetilde{T}_L^b,\widetilde{T}_L^c\}\right]-\sum_R \mathrm{Tr}\left[\widetilde{T}_R^a\{\widetilde{T}_R^b,\widetilde{T}_R^c\}\right]$$

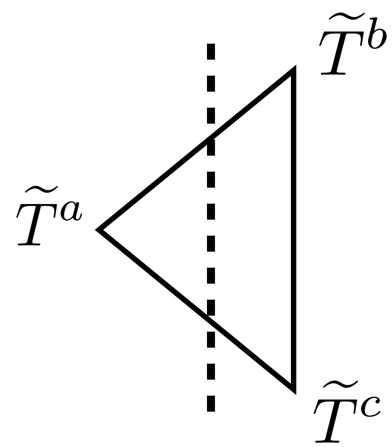
The matching condition survives in the limit $g'\to 0$ when the spectator fermions decouple.

This condition is independent of perturbation theory!

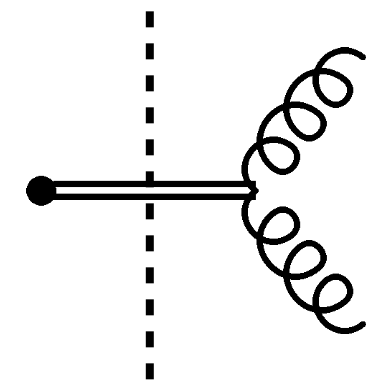
There is however a second possibility: that the global symmetry is **spontaneously broken** at low energies.

In this case there is a second contribution to c_{IR} coming from the massless **Goldstone bosons**

$\mathrm{Im}\,\mathcal{A} = \sum_{m_f=0}$



$+ \sum_{\mathrm{G.B.}}$



In this case, anomaly matching requires

$$\sum_L \mathrm{Tr}\left[T_L^a\{T_L^b,T_L^c\}\right] - \sum_R \mathrm{Tr}\left[T_R^a\{T_R^b,T_R^c\}\right]$$

$$= \sum_L \mathrm{Tr}\left[\widetilde{T}_L^a\{\widetilde{T}_L^b,\widetilde{T}_L^c\}\right] - \sum_R \mathrm{Tr}\left[\widetilde{T}_R^a\{\widetilde{T}_R^b,\widetilde{T}_R^c\}\right] + \text{Goldstone bosons}$$

Applications of anomaly matching: **chiral symmetry breaking**

First example: **QCD with two flavors**

The **global symmetry** group is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B$

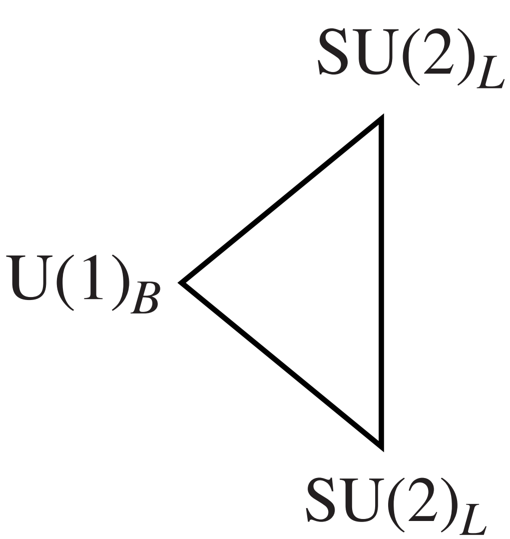
In the UV, the theory contains **two massless quarks** u and d transforming as

$$q_L\text{: } (2,1)_{\frac{1}{3}}$$

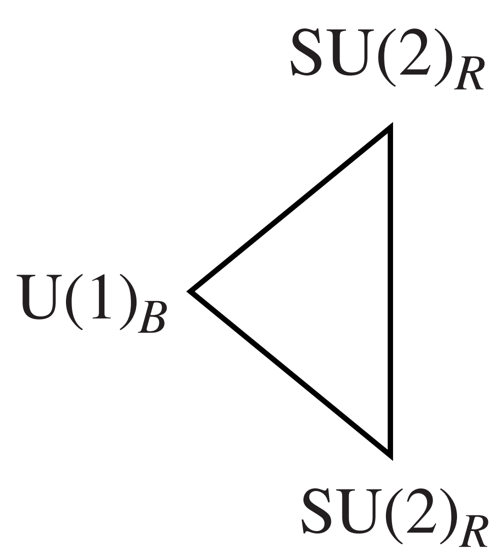
$$q_R\text{: } (1,2)_{\frac{1}{3}}$$

where $(r_L, r_R)_B$

The anomalous triangles in this case are



$$\sim \left(3 \times \frac{1}{3}\right)\delta^{ab} = \delta^{ab}$$



$$\sim -\left(3 \times \frac{1}{3}\right)\delta^{ab} = -\delta^{ab}$$

In the IR we have two possibilities:

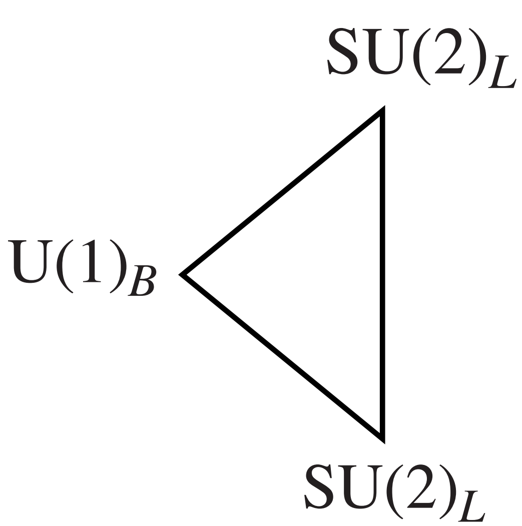
- **Chiral symmetry is unbroken**: we have massless protons and neutrons transforming as

$$N_L: \quad (2,1)_1$$

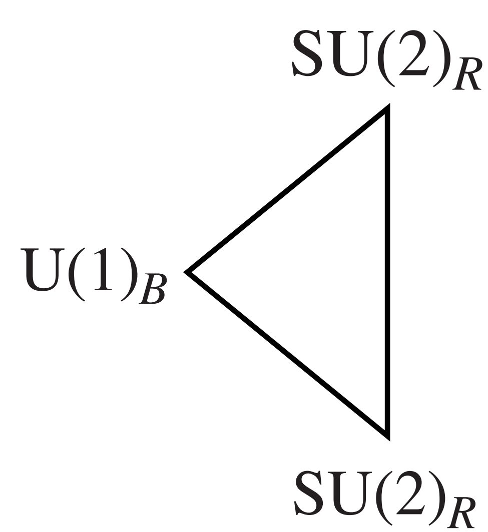
$$N_R: \quad (1,2)_1$$

where again $(r_L, r_R)_B$

Their contribution to the anomaly is



$\sim \delta^{ab}$



$\sim -\delta^{ab}$

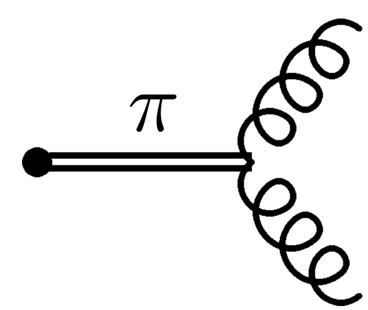
In this scenario UV and IR anomalies **match**.

• **Chiral symmetry is broken** to its vector subgroup

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B \longrightarrow \mathrm{SU}(2)_V \times \mathrm{U}(1)_B$$

protons and neutrons are **massive** and there are three massless pions

In this case, UV anomalies are matched by the **poles** associated with the propagation of the Goldstone boson.



’t Hooft anomaly matching conditions **allows both scenarios**.

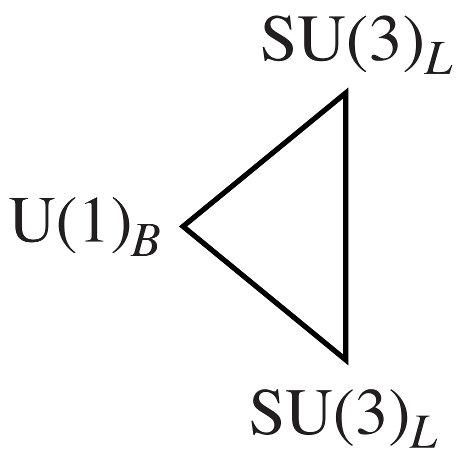
Second example: **QCD with three flavors**

The global symmetry group is now $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_B$

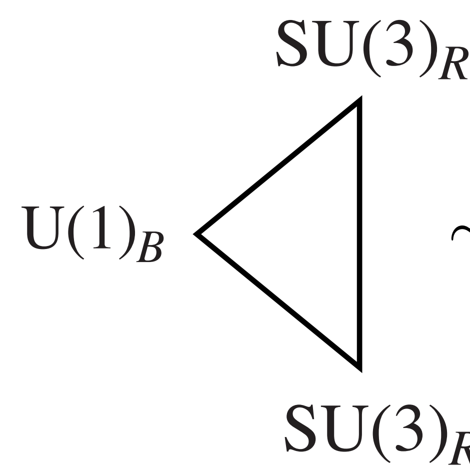
In the **UV**, the massless degrees of freedom are the three **quarks** u, d, and s

$$q_L\text{: } (3,1)_{\frac{1}{3}} \qquad\qquad q_R\text{: } (1,3)_{\frac{1}{3}}$$

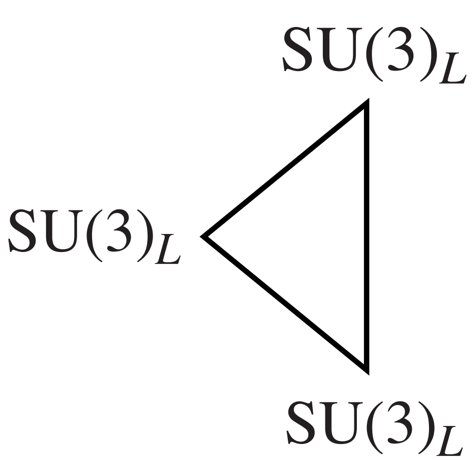
and the anomalies come from the following four triangles:



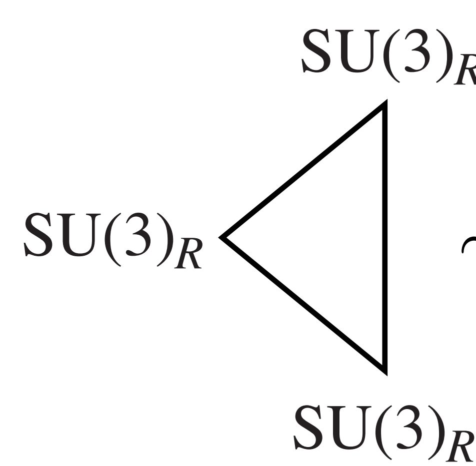
$$\sim \left(3\times\frac{1}{3}\right)\delta^{ab} = \delta^{ab}$$



$$\sim -\left(3\times\frac{1}{3}\right)\delta^{ab} = -\delta^{ab}$$



$$\sim d^{abc}$$



$$\sim -d^{abc}$$

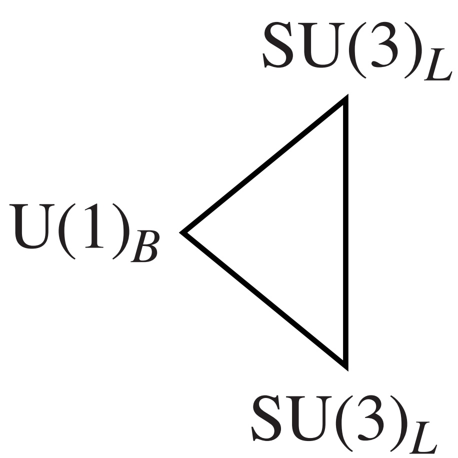
Second example: **QCD with three flavors**

The global symmetry group is now $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_B$

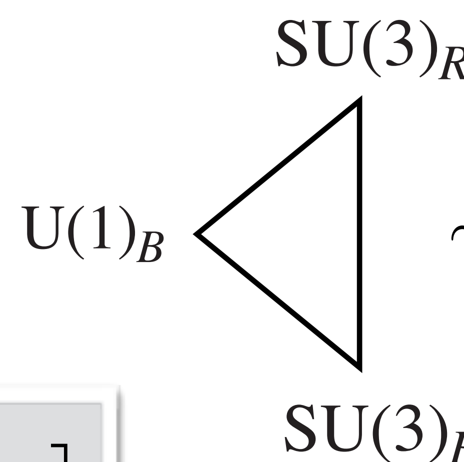
In the **UV**, the massless degrees of freedom are the three **quarks** u, d, and s

$$q_L\text{:}\ (3,1)_{\frac{1}{3}} \qquad\qquad q_R\text{:}\ (1,3)_{\frac{1}{3}}$$

and the anomalies come from the following four triangles:

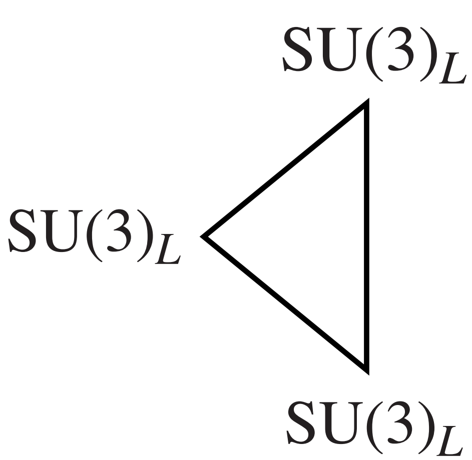


$$\sim \left(3\times\frac{1}{3}\right)\delta^{ab} = \delta^{ab}$$

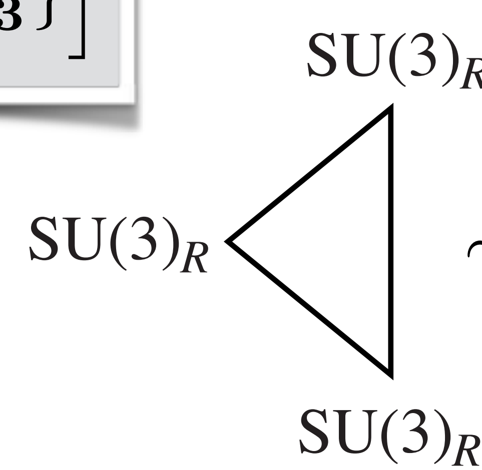


$$\sim -\left(3\times\frac{1}{3}\right)\delta^{ab} = -\delta^{ab}$$

$$d^{abc} \equiv \mathrm{Tr}\left[T^a_{\mathbf{3}}, \{T^b_{\mathbf{3}}, T^c_{\mathbf{3}}\}\right]$$



$$\sim d^{abc}$$



$$\sim -d^{abc}$$

Let us **assume** that chiral symmetry remains **unbroken the IR**. Hence, we have an **octet** of massless baryons

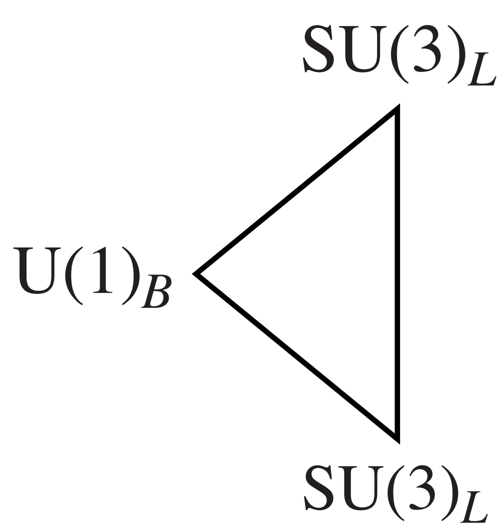
$$p, n, \Sigma^{+}, \Sigma^{-}, \Sigma^{0}, \Lambda, \Xi^{0}, \Xi^{-}$$

transforming as

$$N_L : \ (8, 1)_1$$

$$N_R : \ (1, 8)_1$$

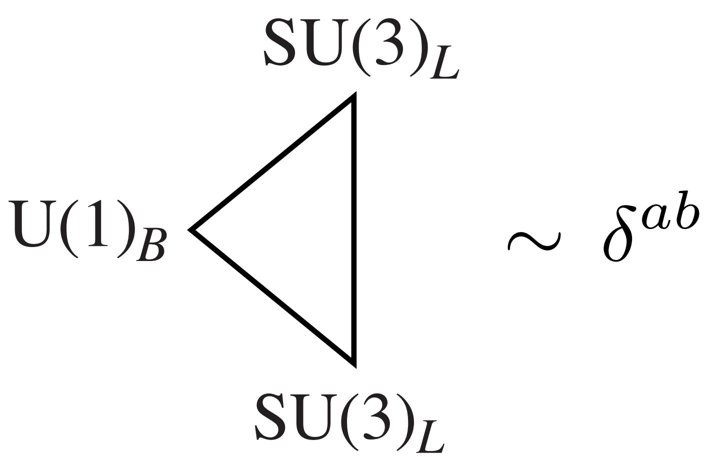
Then, we find for example,



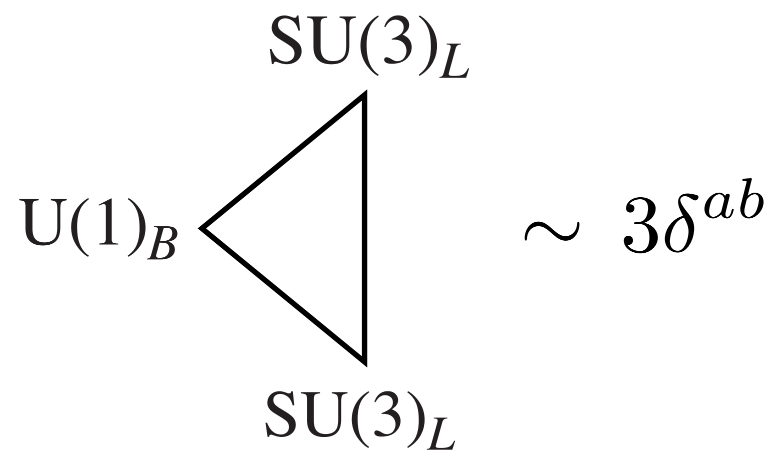
$$\sim \ \mathrm{Tr}\left(T^a_{\mathbf{8}} T^b_{\mathbf{8}}\right) = 3\delta^{ab}$$

Then, **assuming unbroken chiral invariance**, we have found

UV



IR



Anomalies do not match!

This means that QCD with three flavors ***necessarily*** **undergoes chiral symmetry breaking**

$$\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_B \longrightarrow \mathrm{SU}(3)_V \times \mathrm{U}(1)_B$$

and the anomaly is matched by Goldstone boson poles.

By using anomaly matching we obtained **nonperturbative** information of the theory.