Including non-Abelian fields: the singlet anomaly

Instead of QED, we consider now a fermion coupled (in a certain representation) to an external **non-Abelian** gauge theory

$$S = \int d^4x \Big(i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi + g\overline{\psi}T^a_{\mathbf{R}}\gamma^\mu\psi\mathscr{A}^a_\mu \Big)$$

Classically, the gauge current $J^{\mu a}_{\mathrm{V}} = \overline{\psi}\gamma^\mu T^a_{\mathbf{R}}\psi$ satisfies the conservation equation

$$(D_\mu J^\mu_{\mathrm{V}})^a = 0 \quad \longrightarrow \quad \partial_\mu J^{\mu a}_{\mathrm{V}} + g f^{abc}\mathscr{A}^b_\mu J^{\mu c}_{\mathrm{V}} = 0$$

In addition we also have global axial transformations

$$\psi \longrightarrow e^{i\beta\gamma_5}\psi \qquad \overline{\psi} \longrightarrow \overline{\psi}e^{i\beta\gamma_5}$$

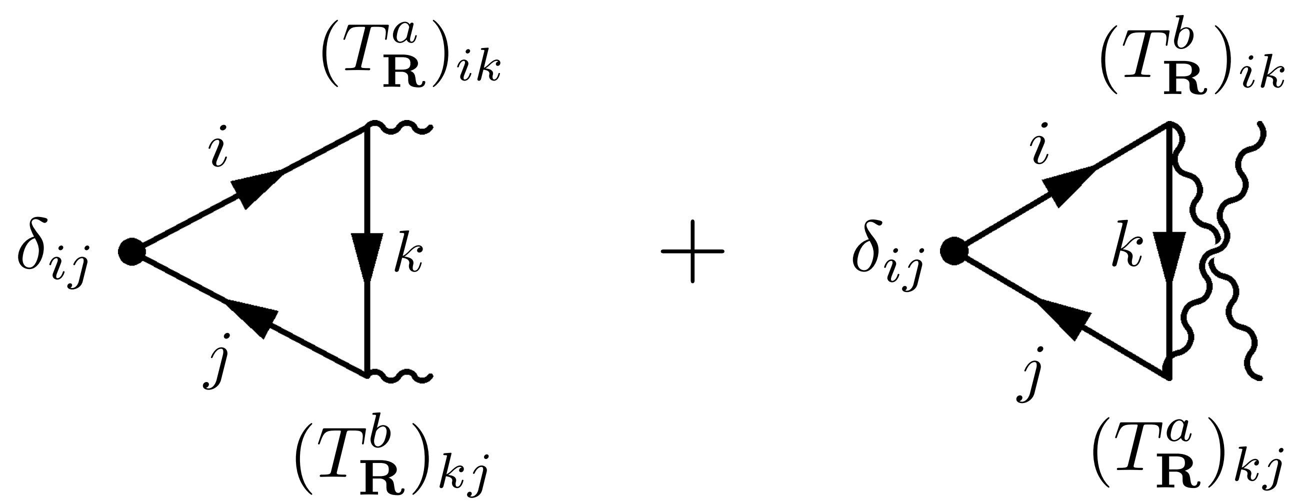
while its associated **singlet** axial current $J^\mu_{\mathrm{A}} = \overline{\psi}\gamma^\mu\gamma_5\psi$ satisfies the identity

$$\partial_\mu J^\mu_{\mathrm{A}} = 2im\overline{\psi}\psi$$

Similarly to QED, the calculation of the axial anomaly boils down to computing

$$\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = -\frac{g^2}{2}\int d^4y_1 d^4y_2 \partial_\mu^{(x)}\langle 0|T[J_{\rm A}^\mu(x)J_{\rm V}^{\alpha a}(y_1)J_{\rm V}^{\beta b}(y_2)]|0\rangle \mathscr{A}_\alpha^a(y_1)\mathscr{A}_\beta^b(y_2)+\ldots$$

Diagrammatically, we have again two triangle diagrams, these time with gauge group generators on the "vector" vertices



The two diagrams share the same color factor

$$\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}}) = \mathrm{Tr}\,(T^b_{\mathbf{R}}T^a_{\mathbf{R}})$$

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -\frac{g^2}{2}\int d^4y_1 d^4y_2 \partial^{(x)}_\mu \langle 0|T[J^\mu_{\rm A}(x)J^{\alpha a}_{\rm V}(y_1)J^{\beta b}_{\rm V}(y_2)]|0\rangle \mathscr{A}^a_\alpha(y_1)\mathscr{A}^b_\beta(y_2) + \ldots$$

The rest of the calculation is identical to the case of QED. In momentum space, we get

$$(p+q)^\mu i\Gamma^{ab}_{\mu\alpha\beta}(p,q) = \frac{ig^2}{2\pi^2}\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}})\epsilon_{\alpha\beta\sigma\nu}p^\sigma q^\nu + 2mi\Gamma^{ab}_{\alpha\beta}(p,q)$$

Adding the external gauge fields and Fourier transforming back to position space, this leads to

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}})\partial_\mu\mathscr{A}^a_\nu\partial_\alpha\mathscr{A}^b_\beta = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}})\partial_\mu\Big(\mathscr{A}^a_\nu\partial_\alpha\mathscr{A}^b_\beta\Big)$$

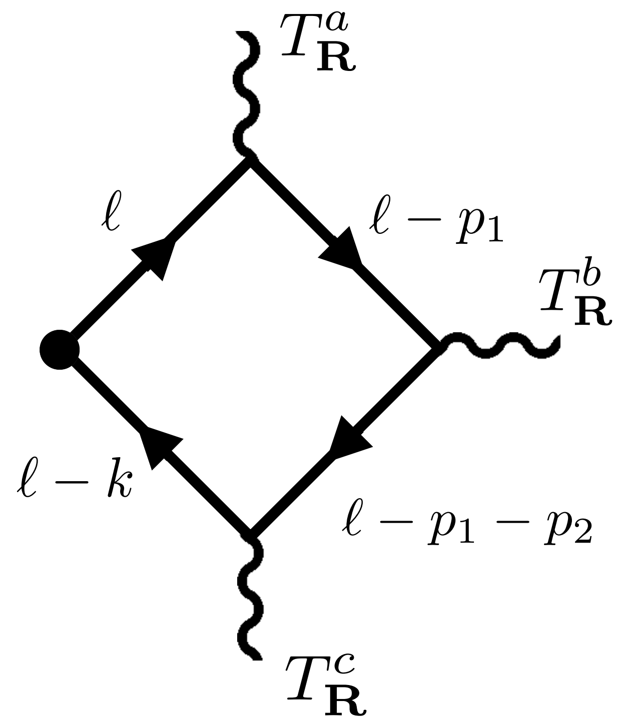
$$\Downarrow$$

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\Big(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta\Big)$$

The problem with this result is that it is **not gauge invariant**!

In fact, in the case of the singlet anomaly the triangle diagram is not enough.

We need to compute the **box diagrams** as well:



$+$ permutations

whose contributions are of the form

$$i\Gamma^{\mu\alpha\beta\gamma}(k,p_1,p_2) = ig^3\text{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}}T^c_{\mathbf{R}})$$

$$\times \int \frac{d^4\ell}{(2\pi)^4}\text{Tr}\left(\gamma^\mu\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}\gamma^\alpha\frac{i}{\not{\ell}-\not{p}_1-\not{p}_2-m+i\epsilon}\gamma^\beta\frac{i}{\not{\ell}-\not{p}_1-m+i\epsilon}\gamma^\sigma\frac{i}{\not{\ell}-m+i\epsilon}\right)$$

$$+ \text{ permutations}$$

In computing the axial-vector Ward identity $k_\mu i\Gamma^{\mu\alpha\beta\gamma}(k,p_1,p_2)$ we encounter the trace

$$\mathrm{Tr}\left(\not{k}\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}\gamma^\alpha\frac{i}{\not{\ell}-\not{p}_1-\not{p}_2-m+i\epsilon}\gamma^\beta\frac{i}{\not{\ell}-\not{p}_1-m+i\epsilon}\gamma^\sigma\frac{i}{\not{\ell}-m+i\epsilon}\right)$$

that we rewrite using

$$\not{k}\gamma_5=\gamma_5(\not{\ell}-\not{k}-m)+(\not{\ell}-m)\gamma_5+2m\gamma_5$$

The first two terms cancel one propagator each, while the last one effectively replaces the axial-vector current by the pseudoscalar bilinear.

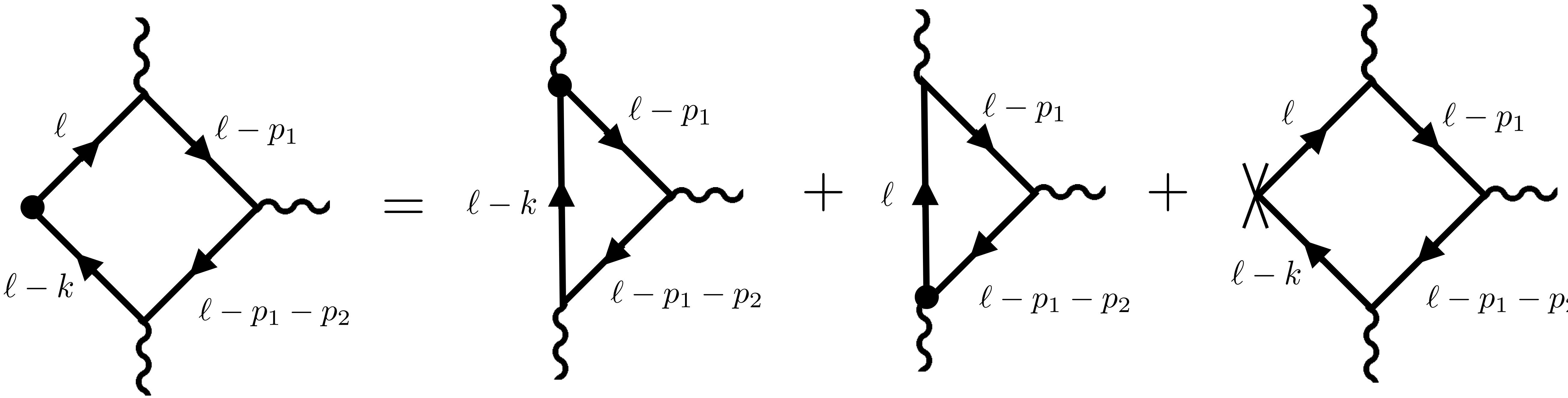
$$\frac{i}{\not{\ell}-m+i\epsilon}\not{k}\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}$$

$$=\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5+\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}+2m\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}$$

$$\frac{i}{\not{\ell} - m + i\epsilon}\not{k}\gamma_5 \frac{i}{\not{\ell} - \not{k} - m + i\epsilon}$$

$$= \frac{i}{\not{\ell} - m + i\epsilon}\gamma_5 + \gamma_5 \frac{i}{\not{\ell} - \not{k} - m + i\epsilon} + 2m\frac{i}{\not{\ell} - m + i\epsilon}\gamma_5 \frac{i}{\not{\ell} - \not{k} - m + i\epsilon}$$

Diagrammatically,



The last term contributes to $2im\langle\overline{\psi}\psi\rangle_{\mathscr{A}}$, whereas the first two "triangles" give corrections to the anomaly **cubic** in the external field.

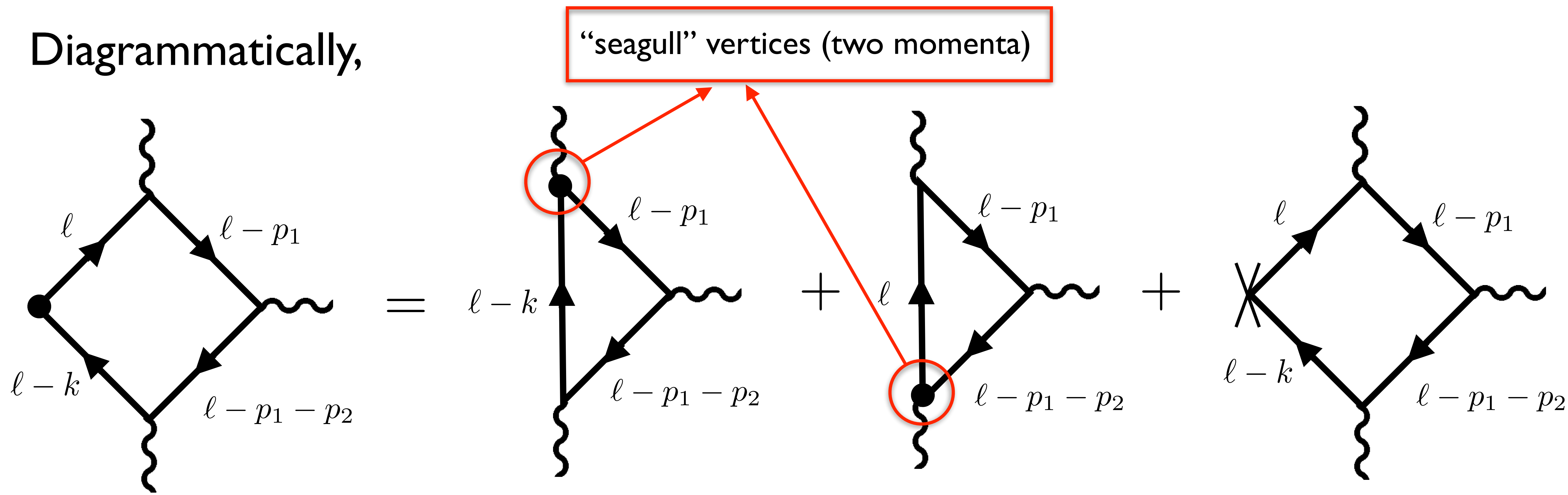
This combines with the triangle diagram to give the **singlet anomaly**:

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu {\rm Tr}\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta + \frac{2}{3}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right)$$

$$\frac{i}{\not\ell - m + i\epsilon}\not k\gamma_5\frac{i}{\not\ell - \not k - m + i\epsilon}$$

$$= \frac{i}{\not\ell - m + i\epsilon}\gamma_5 + \gamma_5\frac{i}{\not\ell - \not k - m + i\epsilon} + 2m\frac{i}{\not\ell - m + i\epsilon}\gamma_5\frac{i}{\not\ell - \not k - m + i\epsilon}$$

Diagrammatically,



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$$\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta + \frac{2}{3}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right)$$

Here we identify the **Chern-Simons form**,

$$\epsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta + \frac{2}{3}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right) = \frac{1}{4}\mathrm{Tr}\left(\mathscr{F}^{\mu\nu}\widetilde{\mathscr{F}}_{\mu\nu}\right)$$

so the singlet anomaly can be written as

$$\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\right)$$

which is **gauge invariant**.

It is important to stress that although there is contribution to the anomaly from the box diagram, its coefficient is determined by the **triangle diagram**

Gauge anomalies

Prelude: quantum symmetries vs. gauge invariance

By Wigner's theorem, **global symmetries** are implemented on the Hilbert space by unitary or antiunitary operators:

$$\mathcal{U}(\alpha_i)|\psi\rangle = |\psi'\rangle \qquad \text{where, generically} \qquad |\psi\rangle \neq |\psi'\rangle$$

As an **example**, let us look at the hydrogen atom: a $\mathrm{SO}(3)$ rotation acts on a state as

$$\mathcal{U}(\theta,\varphi,\psi)|n,j,m\rangle = \sum_{m'=-j}^{j} \mathscr{D}^{(j)}_{mm'}(\theta,\varphi,\psi)|n,j,m'\rangle$$

Gauge invariance is very different from this. In a gauge theory, a physical state is represented by **infinitely many rays** in the Hilbert space.

The space of physical states is smaller than the "naive" Hilbert space of the theory

$$\mathscr{H}_{\text{phys}} = \mathscr{H}/\mathscr{G}$$

Thus, **gauge invariance is not a symmetry but a redundancy**. It is a **technicality** that allows to describe a spin-1 (or spin-2) theory in a way compatible with **locality** and **Lorentz invariance**.

Some of these redundant states, however, have negative norm, e.g.

$$|\Psi\rangle = A_0|\Omega\rangle \quad \longrightarrow \quad \langle\Psi|\Psi\rangle < 0$$

It is thanks to gauge invariance that these redundant states are eliminated from the physical spectrum

$$\delta_{\rm gauge}|\psi\rangle_{\rm phys} = 0$$

Since $\delta_{\rm gauge}A_0 = \dot{\epsilon}(x)$ we have

$$\delta_{\rm gauge}|\Psi\rangle \neq 0 \quad \longrightarrow \quad |\Psi\rangle \text{ is not a physical state}$$

The absence of ghost is preserved in time provided the theory is gauge invariant at the quantum level

$$[\delta_{\rm gauge}, H] = 0$$

which guarantees that

$$\delta_{\rm gauge}|\psi(0)\rangle = 0 \quad \longrightarrow \quad \delta_{\rm gauge}|\psi(t)\rangle = 0$$

i.e., the time evolution of a physical state is a physical state.

However, **if gauge invariant is anomalous** ghosts can be generated by time evolution

⬇

the theory becomes **nonunitary**

⬇

gauge anomalies should be **cancelled** in physical theories at all cost

Where can we expect gauge anomalies?

Since

$$\mathscr{P} : \psi_{R,L} \longrightarrow \psi_{L,R}$$

a parity-invariant theory contains as many right- and left-handed fermions in the same representation.

Thus, we can build **gauge-invariant mass terms** and the theory can be regularized using **Pauli-Villars** fields which preserve gauge invariance.

Gauge anomalies can arise only in **parity-violating** theories.

As a first example, we consider N Dirac fermions with charges Q_i **chirally coupled** to an external electromagnetic field

$$S=\sum_{i=j}^{N}\int d^4x\left[i\overline{\psi}_j\gamma^\mu\partial_\mu\psi_j+Q_i\overline{\psi}_j\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\psi_j\mathscr{A}_\mu\right]$$

This theory has a gauge symmetry

$$\psi_j(x)\longrightarrow\frac{1+\gamma_5}{2}\psi_j(x)+e^{iQ_j\alpha(x)}\frac{1-\gamma_5}{2}\psi_j(x)$$

$$\mathscr{A}_\mu(x)\longrightarrow\mathscr{A}_\mu(x)+\partial_\mu\alpha(x)$$

where the associated conserved current is of the V-A type

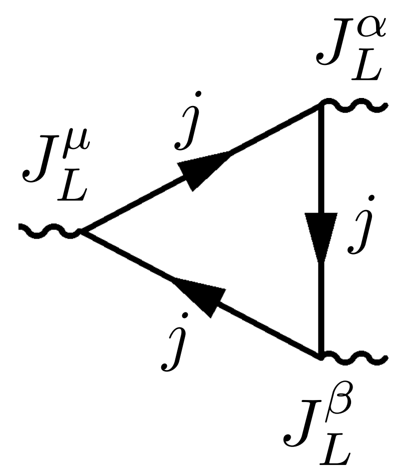
$$J_L^\mu=\sum_{j=1}^{N}Q_j\overline{\psi}_j\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\psi_j\qquad\text{with}\qquad\partial_\mu J_L^\mu=0$$

To study the quantum conservation of the gauge current, we have to compute

$$\partial_\mu \langle J_L^\mu(x)\rangle_{\mathscr{A}} = -\frac{1}{2}\int d^4y_1 d^4y_2 \langle 0|T[J_L^\mu(x)J_L^\alpha(y_1)J_L^\beta(y_2)]|0\rangle \mathscr{A}_\alpha(y_1)\mathscr{A}_\beta(y_2)$$

Diagrammatically, we have to evaluate a triangle diagram with three V-A currents at its vertices

$$\sum_{j=1}^{N}$$



(summing over all fermion species in the loop)

where **Bose symmetry** has to be imposed on **all three vertices**

Even before computing it, we see that the result should be proportional to the quantity

$$\partial_\mu \langle J_L^\mu\rangle_{\mathscr{A}} \sim \sum_{j=1}^{N} Q_j^3$$

To take advantage of our previous calculations, we write

$$J_L^\mu = \frac{1}{2}\Big(J_{\rm V}^\mu - J_{\rm A}^\mu\Big)$$

The anomaly is associated with the parity-violating part of the amplitude that contains the terms

$$\langle 0|T[J_A^\mu J_V^\alpha J_V^\beta]|0\rangle + \langle 0|T[J_V^\mu J_A^\alpha J_V^\beta]|0\rangle + \langle 0|T[J_V^\mu J_V^\alpha J_A^\beta]|0\rangle + \langle 0|T[J_A^\mu J_A^\alpha J_A^\beta]|0\rangle$$

Moving the γ_5 's around, we find that the calculation reduces to the one of the axial anomaly. The final result is:

$$\partial_\mu \langle J_L^\mu(x)\rangle_{\mathscr{A}} = -\frac{1}{96\pi^2}\left(\sum_{j=1}^N Q_j^3\right)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

Gauge invariance is then anomalous unless

$$\sum_{j=1}^N Q_j^3 = 0$$

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Gauge invariance is then anomalous unless

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A similar calculation for a left-handed theory

$$S = \sum_{i=j}^{N} \int d^4x \left[i\overline{\psi}_j \gamma^\mu \partial_\mu \psi_j + \widetilde{Q}_i \overline{\psi}_j \gamma^\mu \left(\frac{1+\gamma_5}{2} \right) \psi_j \mathscr{A}_\mu \right]$$

yields

$$\partial_\mu \langle J_R^\mu(x) \rangle_{\mathscr{A}} = \frac{1}{96\pi^2} \left(\sum_{j=1}^{N} \widetilde{Q}_j^3 \right) \epsilon^{\mu\nu\alpha\beta} \mathscr{F}_{\mu\nu} \mathscr{F}_{\alpha\beta}$$

Finally, for a theory with N_R right-handed and N_L left-handed fermions, the anomaly of the gauge current reads

$$\partial_\mu \langle J^\mu(x) \rangle_{\mathscr{A}} = \frac{1}{96\pi^2} \left(\sum_{j=1}^{N_R} \widetilde{Q}_j^3 - \sum_{j=1}^{N_L} Q_j^3 \right) \epsilon^{\mu\nu\alpha\beta} \mathscr{F}_{\mu\nu} \mathscr{F}_{\alpha\beta}$$

We analyze now the **non-Abelian** case

$$S=\int d^4x\left[i\overline{\psi}\gamma^\mu\Big(\partial_\mu-i\mathcal{L}_\mu\Big)\left(\frac{1-\gamma_5}{2}\right)\psi+i\overline{\psi}\gamma^\mu\Big(\partial_\mu-i\mathcal{R}_\mu\Big)\left(\frac{1+\gamma_5}{2}\right)\psi\right]$$

where we have introduced **external gauge fields** coupled respectively to the **right-** and **left-handed component** of the fermion

$$\mathcal{L}_\mu(x)=\mathcal{L}^a_\mu(x)T^a \qquad\qquad \mathcal{R}_\mu(x)=\mathcal{R}^a_\mu(x)T^a$$

This theory has a $G_L\times G_R$ gauge invariance

$$\psi(x)\longrightarrow e^{i\epsilon^a_L(x)T^a}\left(\frac{1-\gamma_5}{2}\right)\psi(x)+e^{i\epsilon^a_R(x)T^a}\left(\frac{1+\gamma_5}{2}\right)\psi(x)$$

$$\mathcal{L}_\mu(x)\longrightarrow ie^{i\epsilon^a_L(x)T^a}\partial_\mu e^{-i\epsilon^a_L(x)T^a}+e^{i\epsilon^a_L(x)T^a}\mathcal{L}_\mu(x)e^{-i\epsilon^a_L(x)T^a}$$

$$\mathcal{R}_\mu(x)\longrightarrow ie^{i\epsilon^a_R(x)T^a}\partial_\mu e^{-i\epsilon^a_R(x)T^a}+e^{i\epsilon^a_R(x)T^a}\mathcal{R}_\mu(x)e^{-i\epsilon^a_R(x)T^a}$$

Alternatively, we can describe the theory in terms of **vector** and **axial gauge** fields

$$S = \int d^4x \left[i\overline{\psi}\gamma^\mu \Big(\partial_\mu - i\mathcal{V}_\mu - i\mathcal{A}_\mu \gamma_5 \Big) \psi \right]$$

where $\mathcal{V}_\mu = \mathcal{V}_\mu^a T^a$ and $\mathcal{A}_\mu = \mathcal{A}_\mu^a T^a$ are given by

$$\mathcal{V}_\mu = \frac{1}{2}\Big(\mathcal{L}_\mu + \mathcal{R}_\mu \Big) \qquad \mathcal{A}_\mu = \frac{1}{2}\Big(\mathcal{R}_\mu - \mathcal{L}_\mu \Big)$$

In terms of these fields, we have **vector and axial gauge transformations**

$$\psi(x) \longrightarrow e^{i\alpha^a(x)T^a}\psi(x)$$

$$\mathcal{V}_\mu(x) \longrightarrow i e^{i\alpha^a(x)T^a}\partial_\mu e^{-i\alpha^a(x)T^a} + e^{i\alpha^a(x)T^a}\mathcal{V}_\mu(x) e^{-i\alpha^a(x)T^a}$$

$$\mathcal{A}_\mu(x) \longrightarrow e^{i\alpha^a(x)T^a}\mathcal{A}_\mu(x) e^{-i\alpha^a(x)T^a}$$

$$\psi(x) \longrightarrow e^{i\beta^a(x)T^a\gamma_5}\psi(x)$$

$$\mathcal{V}_\mu(x) \longrightarrow e^{i\beta^a(x)T^a}\mathcal{V}_\mu(x) e^{-i\beta^a(x)T^a}$$

$$\mathcal{A}_\mu(x) \longrightarrow i e^{i\beta^a(x)T^a}\partial_\mu e^{-i\beta^a(x)T^a} + e^{i\beta^a(x)T^a}\mathcal{A}_\mu(x) e^{-i\beta^a(x)T^a}$$

A word of **warning**:

Formally, the theory in terms of axial and vector gauge fields seems to have a gauge symmetry

$$G_V \times G_A$$

However, **non-Abelian axial transformations do not close** so they do not define proper gauge invariance

$$e^{i\beta^a T^a \gamma_5} e^{i\beta'^b T^b \gamma_5} = e^{i(\beta^a + \beta'^a) T^a \gamma_5 + \frac{1}{2}\beta^a \beta'^b [T^a, T^b] + \dots}$$

The transformations close only in the Abelian case.

Thus, the only *bona fide* gauge fields of the theory are the ones associated with

$$G_L \qquad G_R \qquad G_V$$

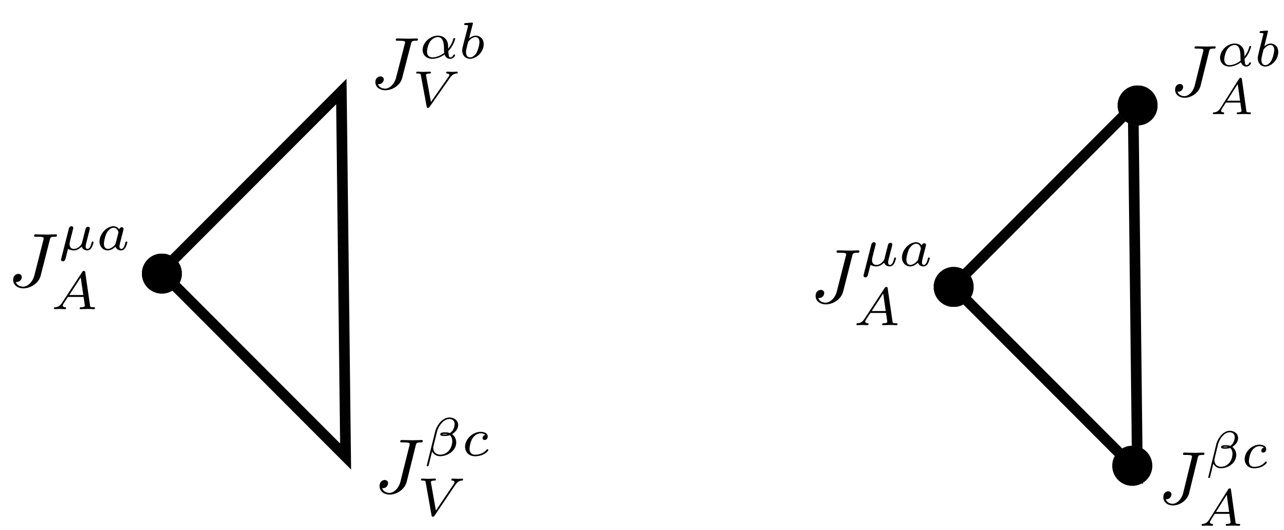
The classical conservation equations for the vector and axial-vector currents are

$$(\mathcal{D}_\mu J^\mu_{\rm A})^a = 0 \quad \longrightarrow \quad \begin{aligned} \partial_\mu J^{\mu a}_{\rm A} + f^{abc}\mathcal{V}^b_\mu J^{\mu c}_{\rm A} + f^{abc}\mathcal{A}^b_\mu J^{\mu c}_{\rm A} &= 0 \\ (D_\mu J^\mu_{\rm A})^a + f^{abc}\mathcal{A}^b_\mu J^{\mu c}_{\rm A} &= 0 \end{aligned}$$

To find the anomaly we have to calculate

$$\langle (\mathcal{D}_\mu J^\mu_{\rm A})^a \rangle_{\mathcal{A},\mathcal{V}} = \frac{1}{Z}\int \mathscr{D}\psi \mathscr{D}\overline{\psi}\Big(\partial_\mu J^{\mu a}_{\rm A} + f^{abc}\mathcal{V}^b_\mu J^{\mu c}_{\rm A} + f^{abc}\mathcal{A}^b_\mu J^{\mu c}_{\rm A}\Big) e^{i\int d^4x [i\overline{\psi}\gamma^\mu(\partial_\mu - i\mathcal{V}_\mu - i\mathcal{A}_\mu\gamma_5)\psi]}$$

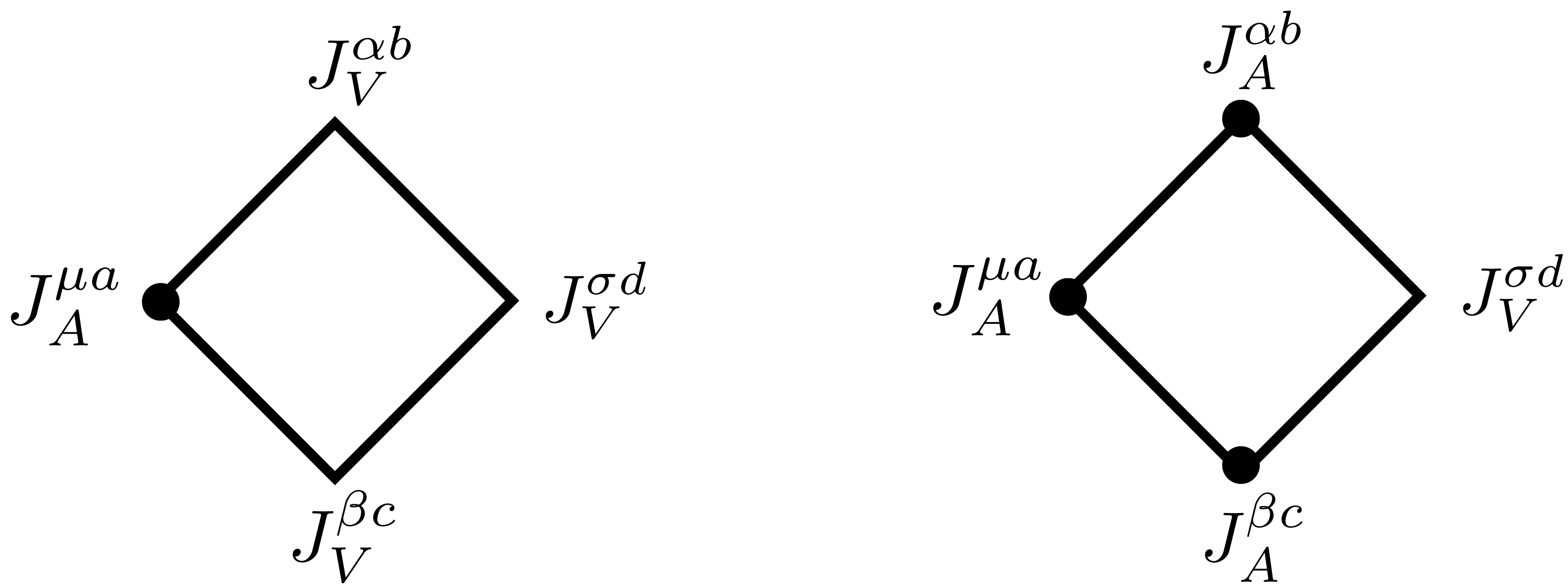
Expanding in perturbation theory, the terms with two gauge fields come as usual from the triangle diagram. The parity-violating ones are



$$\text{Anomaly} = \langle(\partial_\mu J_{\rm A}^{\mu a} + f^{abc}\mathcal{V}_\mu^b J_{\rm A}^{\mu c} + f^{abc}\mathcal{A}_\mu^b J_{\rm A}^{\mu c})\rangle_{\mathcal{V},\mathcal{A}}$$

In the non-Abelian case, there are terms in the triangle with three gauge fields.

Their contribution **combines** with terms coming from the (logarithmically divergent) box diagrams



$$\text{Anomaly} = \langle(\partial_\mu J^{\mu a}_{\mathrm{A}} + f^{abc}\mathcal{V}^b_\mu J^{\mu c}_{\mathrm{A}} + f^{abc}\mathcal{A}^b_\mu J^{\mu c}_{\mathrm{A}})\rangle_{\mathcal{V},\mathcal{A}}$$

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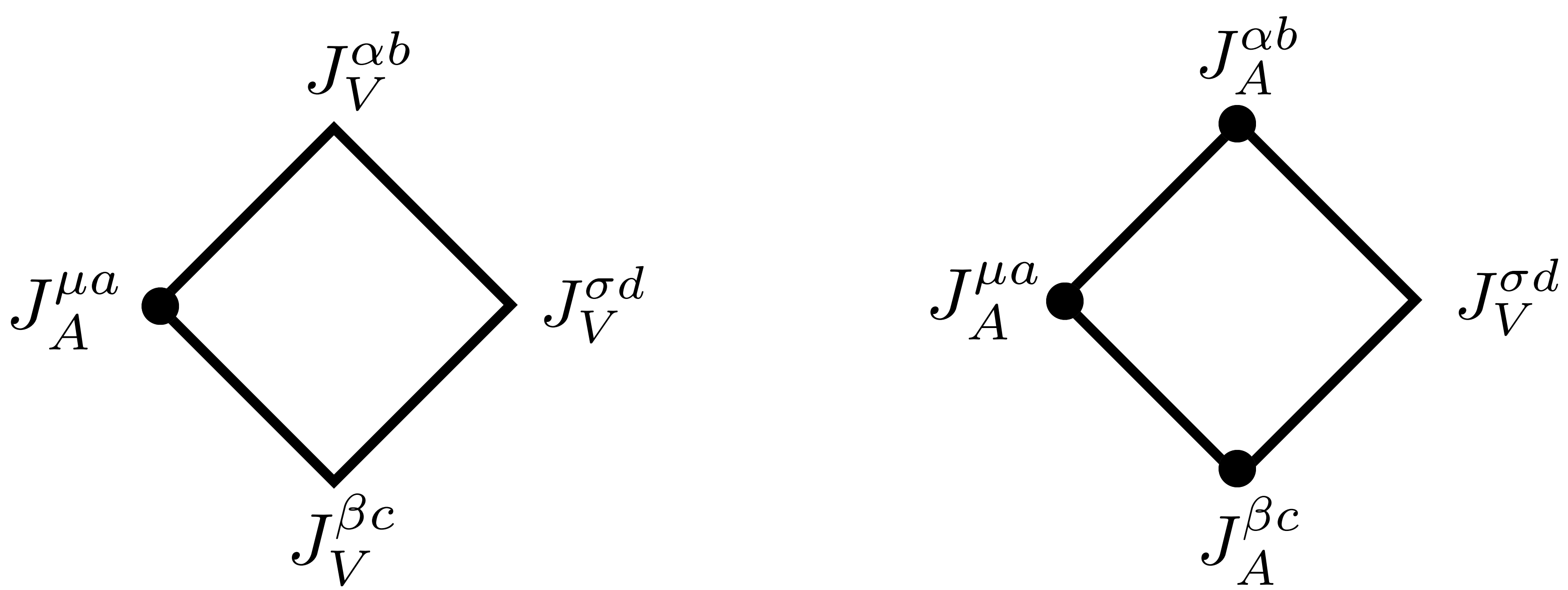

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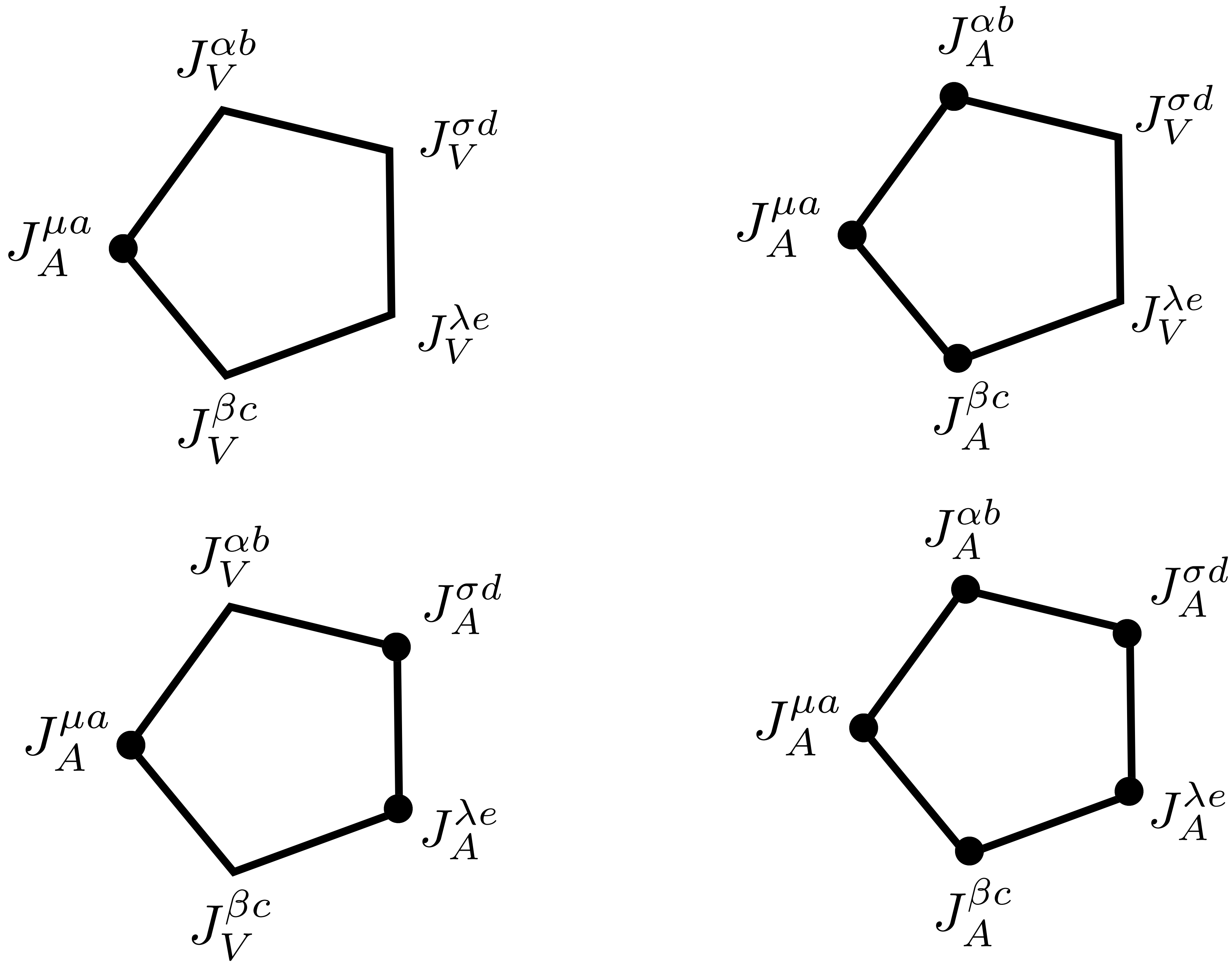
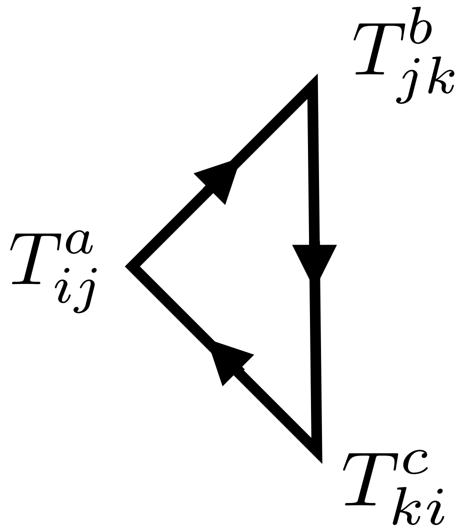
Finally, there are also contributions to the anomaly from the (UV finite) pentagon diagrams:



What about the group theory factors?

For **triangle** we have (AVV and AAA):

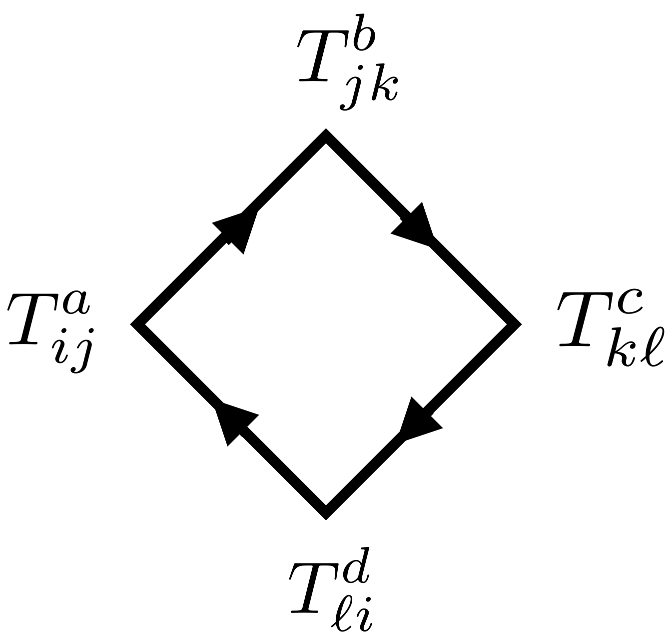


+Bose symmetry

$$\sim \epsilon_{\mu\nu\alpha\beta} p^\mu q^\nu \mathscr{A}^\alpha(p) \mathscr{A}^\beta(q)$$

$$\sim \mathrm{Tr}\left[T^a\{T^b, T^c\}\right]$$

whereas the result for the **box** is (AVVV and AAAV):



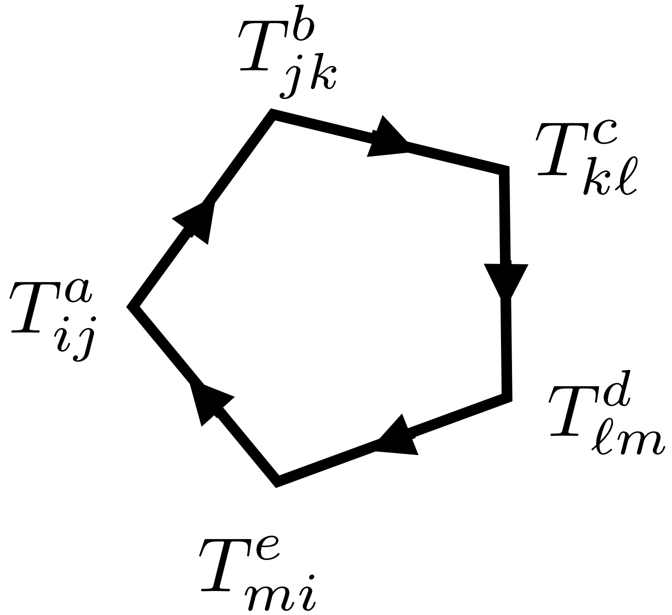
+Bose symmetry

$$\sim \epsilon_{\mu\nu\alpha\beta} p_i^\mu \mathscr{A}^\nu(p_1) \mathscr{A}^\alpha(p_2) \mathscr{A}^\beta(p_3)$$

$$\sim \mathrm{Tr}\left[T^a\{T^b, [T^c, T^d]\}\right]$$

$$= \; i f^{cde} \mathrm{Tr}\left[T^a\{T^b, T^e\}\right]$$

Finally, we deal with the **pentagon** (AVVVV, AVVAA, and AAAA):



$$\sim \epsilon_{\mu\nu\alpha\beta}\mathscr{A}^\mu(p_1)\mathscr{A}^\nu(p_2)\mathscr{A}^\alpha(p_3)\mathscr{A}^\beta(p_4)$$

+Bose symmetry $\longrightarrow$ $\sim \mathrm{Tr}\left[T^a T^{[b}T^c T^d T^{e]}\right]$

$$\sim f^{r[bc}f^{de]s}\mathrm{Tr}\left[T^a\{T^r, T^s\}\right]$$

- The box and pentagon diagrams only contribute to non-Abelian case.

- The cancellation condition for the triangle diagram

$$\mathrm{Tr}\left[T^a\{T^b, T^c\}\right] = 0$$

automatically implies the **cancellation of the box and the pentagon** as well.

Therefore, to cancel the gauge anomaly we only have to care about the triangle!

Computing all these diagrams and imposing vector current conservation

$$\langle(\mathcal{D}_\mu J^\mu_{\rm V})^a\rangle_{\mathcal{V},\mathcal{A}} = 0$$

one arrives at the expression of the **Bardeen anomaly**

$$\langle(\mathcal{D}_\mu J^\mu_{\rm A})^a\rangle_{\mathcal{V},\mathcal{A}} = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left\{T^a\left[\mathcal{V}_{\mu\nu}\mathcal{V}_{\alpha\beta} + \frac{1}{3}\mathcal{A}_{\mu\nu}\mathcal{A}_{\alpha\beta}\right.\right.$$
$$\left.\left. - \frac{8}{3}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{V}_{\alpha\beta} + \mathcal{A}_\mu\mathcal{V}_{\nu\alpha}\mathcal{A}_\beta + \mathcal{V}_{\mu\nu}\mathcal{A}_\alpha\mathcal{A}_\beta\Big) + \frac{32}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\alpha\mathcal{A}_\beta\right]\right\}$$

where

$$\mathcal{V}_{\mu\nu} = \partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu - i[\mathcal{V}_\mu,\mathcal{V}_\nu] - i[\mathcal{A}_\mu,\mathcal{A}_\nu]$$

$$\mathcal{A}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{V}_\mu,\mathcal{A}_\nu] - i[\mathcal{A}_\mu,\mathcal{V}_\nu]$$

The result is **covariant** only with respect vector gauge transformations (it depends on the vector field strength $\mathcal{V}_{\mu\nu}$ alone).

$$\langle (\mathcal{D}_\mu J^\mu_{\rm A})^a \rangle_{\mathcal{V},\mathcal{A}} = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,\left\{T^a\left[\mathcal{V}_{\mu\nu}\mathcal{V}_{\alpha\beta} + \frac{1}{3}\mathcal{A}_{\mu\nu}\mathcal{A}_{\alpha\beta} - \frac{8}{3}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{V}_{\alpha\beta} + \mathcal{A}_\mu\mathcal{V}_{\nu\alpha}\mathcal{A}_\beta + \mathcal{V}_{\mu\nu}\mathcal{A}_\alpha\mathcal{A}_\beta\Big) + \frac{32}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\alpha\mathcal{A}_\beta\right]\right\}$$

This expression can be used as a "**master formula**" for different situations.

- **QED axial anomaly**: Abelian case, $\mathcal{A}_\mu = 0, \mathcal{V}_\mu = e\mathscr{A}_\mu$

$$\partial_\mu \langle J^\mu_{\rm A}\rangle_{\mathscr{A}} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

- **Nonabelian singlet anomaly**: $T^a \longrightarrow \mathbb{I}, \mathcal{A}_\mu = 0, \mathcal{V}_\mu = g\mathscr{A}_\mu$

$$\partial_\mu \langle J^\mu_{\rm A}\rangle_{\mathscr{A}} = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\right)$$

$$\langle (\mathcal{D}_\mu J^\mu_{\rm A})^a \rangle_{\mathcal{V},\mathcal{A}} = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left\{T^a\left[\mathcal{V}_{\mu\nu}\mathcal{V}_{\alpha\beta} + \frac{1}{3}\mathcal{A}_{\mu\nu}\mathcal{A}_{\alpha\beta} - \frac{8}{3}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{V}_{\alpha\beta} + \mathcal{A}_\mu\mathcal{V}_{\nu\alpha}\mathcal{A}_\beta + \mathcal{V}_{\mu\nu}\mathcal{A}_\alpha\mathcal{A}_\beta\Big) + \frac{32}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\alpha\mathcal{A}_\beta\right]\right\}$$

This expression can be used as a "master formula" for different situations.

- **"Right-handed" QED**: Abelian case, $T^a \longrightarrow Q$

$$\mathcal{A}_\mu = -\mathcal{V}_\mu = -\frac{1}{2}\mathcal{L}_\mu \qquad J^\mu_L = \frac{1}{2}\Big(J^\mu_{\rm V} - J^\mu_{\rm A}\Big)$$

$$\longrightarrow \quad \begin{cases} \mathcal{V}_\mu = \dfrac{Q}{2}\mathscr{A}_\mu \\ \mathcal{A}_\mu = -\dfrac{Q}{2}\mathscr{A}_\mu \end{cases}$$

$$\partial_\mu\langle J^\mu_L(x)\rangle_{\mathscr{A}} = -\frac{1}{96\pi^2}\left(\sum_{j=1}^N Q_j^3\right)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

We have seen that a theory of a chiral fermion is free of gauge anomalies whenever they transform in a representation $\mathbf{R}$ satisfying

$$d_{\mathbf{R}}^{abc} \equiv \mathrm{Tr}\left[T_{\mathbf{R}}^{a}\{T_{\mathbf{R}}^{b}, T_{\mathbf{R}}^{c}\}\right] = 0 \qquad \textbf{(anomaly coefficients)}$$

Let us do some **group theory**…

A Lie algebra representation is **real** or **pseudoreal** if there is an intertwining operator S satisfying

$$T_{\mathbf{R}}^{a\,*} = -S T_{\mathbf{R}}^{a} S^{-1} \qquad \begin{cases} S^T = S & \text{real} \\ S^T = -S & \text{pseudoreal} \end{cases}$$

Then

$$\mathrm{Tr}\left[T_{\mathbf{R}}^{a}\{T_{\mathbf{R}}^{b}, T_{\mathbf{R}}^{c}\}\right] = \mathrm{Tr}\left[T_{\mathbf{R}}^{a}\{T_{\mathbf{R}}^{b}, T_{\mathbf{R}}^{c}\}\right]^{T} = \mathrm{Tr}\left[(T_{\mathbf{R}}^{a})^{*}\{(T_{\mathbf{R}}^{b})^{*}, (T_{\mathbf{R}}^{c})^{*}\}\right]$$

and for **real** and **pseudoreal** representations

$$\mathrm{Tr}\left[(T_{\mathbf{R}}^{a})^{*}\{(T_{\mathbf{R}}^{b})^{*}, (T_{\mathbf{R}}^{c})^{*}\}\right] = -\mathrm{Tr}\left[S T_{\mathbf{R}}^{a} S^{-1}\{S T_{\mathbf{R}}^{b} S^{-1}, S T_{\mathbf{R}}^{c} S^{-1}\}\right] = -\mathrm{Tr}\left[T_{\mathbf{R}}^{a}\{T_{\mathbf{R}}^{b}, T_{\mathbf{R}}^{c}\}\right]$$

Thus, **real and pseudoreal** are **anomaly-free** representations

$$\mathrm{Tr}\left[T^a_{\mathbf{R}}\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\}\right]=0 \qquad \text{for } \mathbf{R} \text{ real or pseudoreal}$$

This happens for **all** representations of the following groups

- SU(2)
- SO(2N+1)
- SO(4N) for $N \geq 2$
- Sp(2N) for $N \geq 3$
- and the exceptional groups $G_2,\ F_4,\ E_7,\ E_8$

Other groups whose representations are **neither real or pseudoreal** but are still **safe** are

- SO(4N+2) for $N \geq 2$
- E_6

In addition, the **adjoint** representation of any group is real and therefore **safe**.

Potentially dangerous Lie group are

- U(1).
- SU(N) for $N \geq 3$.

In the case of **non-safe groups**, anomalies can be **elliminated** either by choosing an **anomaly free representation** or **summing the contribution of all chiral fields**.

For example, if a theory contains a number of **right- and left-handed fermions** transforming in representations T_+^a and T_-^a the anomaly cancellation condition reads:

$$\sum_{\text{right-handed}} \mathrm{Tr}\left[T_+^a\{T_+^b, T_+^c\}\right] - \sum_{\text{left-handed}} \mathrm{Tr}\left[T_-^a\{T_-^b, T_-^c\}\right] = 0$$

If the gauge group is a direct product, $G_1 \otimes \ldots \otimes G_n$, there might be **mixed gauge anomalies** associated with triangles with "different group factors" at each vertex